STATS 325 Exam Solutions Semester 2, 2013

1a) Consider

$$U = \begin{cases} 1 & \text{with prob } 5/10 \\ 1+U^* & \text{w.p. } 1/10 \text{ where } U^* \sim U \\ 1+U'+U'' & \text{w.p. } 4/10 \text{ where } U' \sim U'' \sim U \text{ and } U', U'' \text{ are independent.} \end{cases}$$

So $H_u(s) = \frac{5}{10}s + \frac{1}{10}s\,\mathbb{E}(s^{U^*}) + \frac{4}{10}s\,\mathbb{E}(s^{U'})\mathbb{E}(s^{U''})$ by independence of U', U''

$\Rightarrow H_u(s) = \frac{5}{10}s + \frac{1}{10}s\,H_u(s) + \frac{4}{10}s\,[H_u(s)]^2$ because $U^* \sim U' \sim U'' \sim U$.

$\Rightarrow 4s\,H_u(s)^2 + (s-10)H_u(s) + 5s = 0$

$$\Rightarrow \quad H_u(s) = \frac{10-s \pm \sqrt{s^2-20s+100-4*20s^2}}{8s}$$

$$\therefore \quad \boxed{H_u(s) = \frac{10-s \pm \sqrt{100-20s-79s^2}}{8s}} \quad \text{as stated.}$$

b) $H_u(s)$ is continuous as $s \downarrow 0$.
$\therefore$ We require $\lim_{s\downarrow 0} H_u(s) = H_u(0) = \mathbb{P}(U=0) = 0$, because at least one step is needed to move 0 to 1.

Consider the (+) root:

$$\lim_{s\downarrow 0}\left\{\frac{10-s+\sqrt{100-20s-79s^2}}{8s}\right\} = \lim_{s\downarrow 0}\left\{\frac{10+\sqrt{100}}{8s}\right\} \neq 0.$$

So the (+) root is impossible.

c) U is defective $\Leftrightarrow H_u(1) < 1$.
Now $H_u(1) = \frac{10-1-\sqrt{100-20-79}}{8} = 1$.

1c cont.) So U is not defective.

Thus $\mathbb{P}(\text{reach state } 1 \mid \text{start at state } 0) = 1$.

1d) $\mathbb{P}(\text{reach state } -1 \mid \text{start at state } 0) = H_v(1) \quad (\mathbb{P}(V<\infty))$

$$= \frac{10-1-\sqrt{100-20-79}}{10}$$

$$= 0.8.$$

e) $\mathbb{P}(\text{return to } 0 \mid \text{start at } 0) = G(1) \quad (\mathbb{P}(T<\infty))$

Now

$$T = \begin{cases} 1 & \text{w.p. } 1/10 \\ 1+U & \text{w.p. } 4/10 \\ 1+V & \text{w.p. } 5/10 \end{cases}$$

So $G(s) = \mathbb{E}(s^T) = \frac{1}{10}s + \frac{4}{10}s\,H_u(s) + \frac{5}{10}s\,H_v(s)$

$\Rightarrow G(1) = \frac{1}{10} + \frac{4}{10}H_u(1) + \frac{5}{10}H_v(1)$

$= \frac{1}{10} + \frac{4}{10} * 1 + \frac{5}{10} * 0.8$ by (c), (d)

$\therefore G(1) = 0.9$ Return probability.

2) a) $\mu > 1$ if the curve $t = G(s)$ is steeper than the line $t = s$ at $s = 1$.
So $\mu > 1$ in processes A, B.

b) $\gamma = 1$ if the curves $t = G(s)$ and $t = s$ only cross at $s = 1$.
So $\gamma = 1$ for curve C only.

c) We require $\gamma = G(\gamma)$ for extinction probability γ.

So $\gamma = p_0 + p_1\gamma + p_2\gamma^2 + \dots$ where $p_k = \mathbb{P}(Y = k)$

Given that $p_0 = \mathbb{P}(Y=0) = \gamma$,

then $\gamma = \gamma + p_1\gamma + p_2\gamma^2 + \dots$

$\Rightarrow (p_1 + p_2\gamma + \dots)\gamma = 0$

In fact there are two solutions: $\gamma = 0$
or $p_1 = p_2 = \dots = 0$
which would imply $p_0 = \gamma = 1$.

So it $\underline{\text{is}}$ possible for $\gamma = \mathbb{P}(Y=0) = 1$.
Thus (c) is FALSE.

d) $Y \sim \text{Bin}(2, \frac{3}{4})$

$$\Rightarrow G(s) = \mathbb{E}(s^Y) = \sum_{y=0}^{2} \binom{2}{y} \left(\frac{3}{4}\right)^y \left(\frac{1}{4}\right)^{2-y} s^y$$
$$= \sum_{y=0}^{2} \binom{2}{y} \left(\frac{3s}{4}\right)^y \left(\frac{1}{4}\right)^{2-y}$$
$$= \left(\frac{3}{4}s + \frac{1}{4}\right)^2 \text{ by Binomial Thm.}$$

2d cont) So $\underline{\underline{G(s) = \frac{1}{16}(3s+1)^2}}$ as stated.

e) $\mathbb{P}(Z_2 = 0) = G_2(0)$
$$= G(G(0))$$
$$= G\left(\frac{1}{16}\right)$$
$$= \frac{1}{16}\left\{\frac{3}{16} + 1\right\}^2$$
$$= \frac{361}{4096}$$
$$= \underline{\underline{0.088.}} \text{ as stated.}$$

f) γ satisfies $G(s) = s$

$$\Rightarrow \frac{1}{16}(3s+1)^2 = s$$
$$9s^2 + 6s + 1 = 16s$$
$$9s^2 - 10s + 1 = 0$$
$$(9s-1)(s-1) = 0$$
$$\Rightarrow \underline{s = \tfrac{1}{9} \text{ or } s = 1.}$$

γ is smallest solution ≥ 0, so $\underline{\underline{\gamma = \frac{1}{9}}}$.

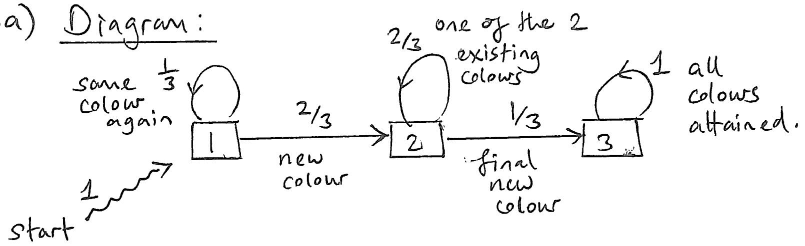
g) All 3 lines must go extinct, so $\underline{\underline{\mathbb{P}(\text{extinction} \mid Z_2 = 3) = \gamma^3 = \frac{1}{729}}}$
(0.00137).

h) Now $(T = \infty) \Leftrightarrow$ (Extinction does not occur),
So $\underline{\underline{T \text{ is defective}}}$ with $\underline{\underline{\mathbb{P}(T = \infty) = 1 - \gamma = \frac{8}{9}}}$.

Then $\mathbb{P}(2 < T < \infty) = 1 - \mathbb{P}(T \leq 2) - \mathbb{P}(T = \infty) = 1 - \mathbb{P}(Z_2 = 0) - \frac{8}{9} = \underline{\underline{0.023.}}$

0.088 by (e)

3a) Diagram:



i) Statespace, $S = \{1, 2, 3\}$.

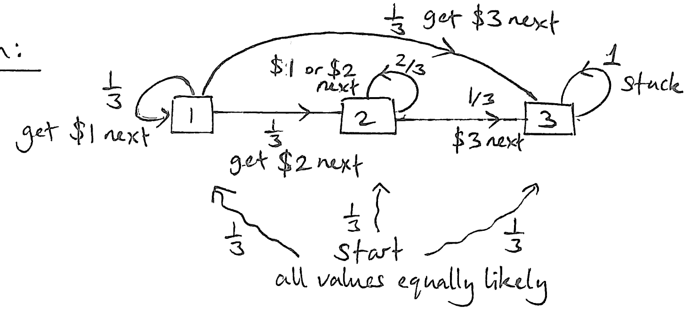
ii) Matrix, $P = P_C$.

iii) $\underset{\sim}{\alpha} = (1, 0, 0)$ because 1 colour is always attained in 1st bar.

iv) Chain is aperiodic but NOT irreducible, as state 3 is a closed class. However, by inspection, the chain does converge to equilibrium because it gets stuck in state 3 with prob 1, from any start state.

v) $\underset{\sim}{\pi} = (0, 0, 1)$.

3b) Diagram:



i) State space, $S = \{1, 2, 3\}$

ii) Matrix, $P = P_B$

iii) $\underset{\sim}{\alpha} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ as all vouchers are equally likely at time $t=1$.

iv) As in (a), aperiodic but not irreducible; but convergence IS guaranteed by inspection.

v) $\underset{\sim}{\pi} = (0, 0, 1)$.

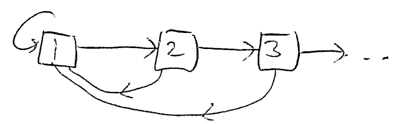
3c) At time $t+1$ we retain same colour w.p. $1/3$ (so $X_{t+1} = X_t + 1$) or switch to new colour w.p. $2/3$ (so $X_{t+1} = 1$).

i) State space, $S = \{1, 2, 3, \ldots\}$

ii) Matrix, $P = P_D$.

iii) $X_1 = 1$ w.p. 1, so $\underset{\sim}{\alpha} = (1, 0, 0, \ldots)$

iv) v) Diagram:



Chain is irreducible;
Aperiodic because of loop on state 1;
So chain does converge to equilibrium, assuming $\underset{\sim}{\pi}$ exists.

Equilibrium eqns: assume $\underset{\sim}{\pi}$ exists, then:

$$\underset{\sim}{\pi}^T P_D = (\pi_1 \; \pi_2 \; \pi_3 \cdots) \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & \cdots \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} & \cdots \\ \vdots & & & & \end{pmatrix} = (\pi_1 \; \pi_2 \; \pi_3 \cdots)$$

Require $\pi_1 + \pi_2 + \ldots = 1$ ①

$\frac{2}{3}(\pi_1 + \pi_2 + \ldots) = \pi_1$ ②

Subst ① in ② $\Rightarrow \frac{2}{3} * 1 = \pi_1$ so $\underline{\underline{\pi_1 = \frac{2}{3}}}$

Continuing: $\frac{1}{3}\pi_1 = \pi_2 \Rightarrow \underline{\underline{\pi_2 = \frac{2}{3} * \frac{1}{3}}}$

$\frac{1}{3}\pi_2 = \pi_3 \Rightarrow \underline{\underline{\pi_3 = \frac{2}{3} * \left(\frac{1}{3}\right)^2}}$

etc: in general, $\frac{1}{3}\pi_n = \pi_{n+1}$ and $\underline{\underline{\pi_n = \frac{2}{3}\left(\frac{1}{3}\right)^{n-1} \quad n = 1, 2, \ldots}}$

3c cont) Check $\sum_{n=1}^{\infty} \pi_n = 1$:

$$\text{LHS} = \frac{2}{3}\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1} = \frac{2}{3}\sum_{m=0}^{\infty}\left(\frac{1}{3}\right)^{m}$$

$$= \frac{2}{3}\cdot\frac{1}{1-1/3} \quad \text{Geometric series}$$

$$= 1$$

$$= \text{RHS}.$$

So the equilibrium distn does exist,

so $\{X_t\}$ does converge and $\underset{\sim}{\pi}^T = \left(\frac{2}{3}, \frac{2}{3}*\frac{1}{3}, \frac{2}{3}\left(\frac{1}{3}\right)^2, \frac{2}{3}\left(\frac{1}{3}\right)^3, \dots\right)$

ie. $\pi_k = \frac{2}{3}\left(\frac{1}{3}\right)^{k-1}$ for $k=1,2,\dots$

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4) a) $\mathbb{P}(X>x) = \int_x^{\infty} \lambda e^{-\lambda u}\,du = \left[-e^{-\lambda u}\right]_x^{\infty} = e^{-\lambda x}$ for $x>0$.

b) $$\mathbb{P}(X>Y) = \mathbb{E}_Y\{\mathbb{P}(X>Y\mid Y)\}$$

$$= \int_0^{\infty} \mathbb{P}(X>Y\mid Y=y)\, f_Y(y)\,dy$$

$$= \int_0^{\infty} e^{-\lambda y}\cdot \mu e^{-\mu y}\,dy \quad \text{using (a)}$$

$$= \mu\int_0^{\infty} e^{-(\lambda+\mu)y}\,dy$$

$$= \mu\left[-\frac{1}{\lambda+\mu}e^{-(\lambda+\mu)y}\right]_0^{\infty}$$

$\therefore$ $\mathbb{P}(X>Y) = \dfrac{\mu}{\lambda+\mu}$ as stated.

4c) Consider: If $X>Y$ then Y happens first, ie a departure happens before an arrival.

So queue moves down, one place to left, assuming state is 1, 2, 3, 4.

Thus $\frac{3}{7} = \frac{\mu}{\lambda+\mu}$ by (b)

$$\frac{3}{7} = \frac{\mu}{8+\mu}$$

$$\Rightarrow 24 + 3\mu = 7\mu$$

$$24 = 4\mu$$

$$\mu = 6.$$

d) Define $m_x = \mathbb{E}(\text{time to reach state } 0 \mid \text{start in state } x)$.

Only the arrows accrue time.

$$m_0 = 0$$

$$m_1 = \frac{3}{7}*\frac{1}{14} + \frac{4}{7}*\left(\frac{1}{14}+m_2\right) = \frac{1}{14} + \frac{4}{7}m_2$$

$$m_2 = \frac{1}{14} + \frac{3}{7}m_1 + \frac{4}{7}m_3$$

$$m_3 = \frac{1}{14} + \frac{3}{7}m_2 + \frac{4}{7}m_4$$

$$m_4 = \frac{1}{6} + m_3$$

Solve for m_4 (final answer).

e) $$P = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 3/7 & 0 & 4/7 & 0 & 0 \\ 0 & 3/7 & 0 & 4/7 & 0 \\ 0 & 0 & 3/7 & 0 & 4/7 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Classes: $\{0,1,2,3,4\}$ only. closed.

4f) P is irreducible, but it is periodic: all states have period $=2$.
Therefore $\{X_t\}$ can not converge to a single equilibrium distn irrespective of start state as $t \to \infty$.

5a) $$\begin{aligned} H(s) &= \mathbb{E}(s^T) \\ &= \mathbb{E}\left(s^{X_1 + \dots + X_N}\right) \\ &= \mathbb{E}_N\left\{ \mathbb{E}\left(s^{X_1 + \dots + X_N} \mid N\right)\right\} \\ &= \mathbb{E}_N\left\{ \mathbb{E}(s^X)^N \right\} \end{aligned}$$ because $X_1 \sim X_2 \sim \dots \sim X$ and $X_1, X_2, \dots$ are indept of each other and of N.

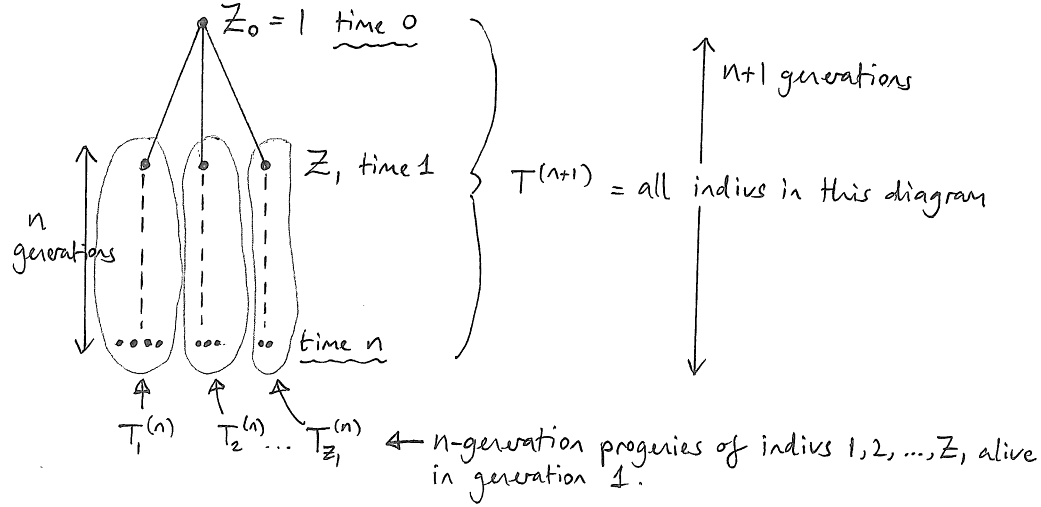
$$= \mathbb{E}_N\left\{ G_X(s)^N \right\}$$

$\therefore$ $\underline{\underline{H(s) = G_N\left(G_X(s)\right)}}$ as stated.

b) $$\begin{aligned} \mathbb{E}(s^Y) &= \sum_{y=0}^{\infty} p q^y s^y \\ &= p \sum_{y=0}^{\infty} (qs)^y \\ &= \frac{p}{1-qs} \quad \text{for } |qs| < 1 \end{aligned}$$

$\Rightarrow$ $\underline{\underline{G(s) = \mathbb{E}(s^Y) = \frac{p}{1-qs} \text{ for } |s| < \frac{1}{q}}}$ as required.

5c) Consider the $n+1$ generations from time 0 to time $n+1$:



Clearly, $$\underbrace{T^{(n+1)}}_{\substack{(n+1)\text{-generation total from time 0} \\ \text{to time } n}} = \underbrace{1}_{\substack{1 \text{ indiv} \\ \text{at time 0}}} + \underbrace{T_1^{(n)} + T_2^{(n)} + \dots + T_{Z_1}^{(n)}}_{\substack{n\text{-generation totals from} \\ \text{each indiv } 1,2,\dots,Z_1 \text{ alive at time 1}}}$$

Where $T_1^{(n)}, T_2^{(n)}, \dots T_{Z_1}^{(n)}$ are independent and $T_1^{(n)} \sim T_2^{(n)} \sim \dots \sim T^{(n)}$.

So $$\begin{aligned} H_{n+1}(s) = \mathbb{E}\left(s^{T^{(n+1)}}\right) &= \mathbb{E}\left(s^{1 + T_1^{(n)} + \dots + T_{Z_1}^{(n)}}\right) \\ &= s\,\mathbb{E}\left(s^{T_1^{(n)} + \dots + T_{Z_1}^{(n)}}\right) \end{aligned}$$ Randomly stopped sum as in (a): $N \rightsquigarrow Z_1$, $X_i \rightsquigarrow T_i^{(n)}$

[using notation & result from (a)] $$= s\, G_{Z_1}\left(G_{T^{(n)}}(s)\right)$$

$\Rightarrow$ $\underline{\underline{H_{n+1}(s) = s\, G\left(H_n(s)\right)}}$ as required. noting $G_{Z_1} \equiv G$, $G_{T^{(n)}} \equiv H_n$.

Valid for $n = 1, 2, 3, \dots$.

5d) $H(s) = s\,G(H(s)) \Rightarrow H(s) = s \cdot \frac{p}{1 - qH(s)}$ by (b)

$$\Rightarrow q\,H(s)^2 - H(s) + ps = 0.$$

5d cont.) So

$$H(s) = \frac{1 \pm \sqrt{1-4pqs}}{2q}$$

Now $H(0) = \mathbb{P}(T=0) = 0$ because there is 1 indiv in gen. 0.

So (+) root giving $\frac{1+1}{2q} = \frac{1}{q}$ cannot be correct,

but (−) root gives 0.

So $H(s) = \frac{1-\sqrt{1-4pqs}}{2q}$.

i) If $p = q = 0.5$ then $H(1) = \frac{1-\sqrt{1-1}}{1} = 1$.

So T is not defective if $p = 0.5$.

Thus T is necessarily finite: $\mathbb{P}(T<\infty) = 1$

$\Rightarrow$ extinction is definite as the process must be finite.

$\therefore \gamma = 1$ when $p = 0.5$.

ii) If $p = 0.4$ then $H(1) = \frac{1-\sqrt{1-4*0.4*0.6}}{2*0.6} = \frac{2}{3}$.

So T is defective if $p = 0.4$.

So $\mathbb{P}(\text{extinction}) = \gamma = \mathbb{P}(T<\infty)$

$= H(1)$

$\therefore \gamma = \frac{2}{3}$ Also solves $G(\gamma) = \gamma$: checked.

6a) Consider $\{N \geq n\} = \{N = n\} \cup \{N \geq n+1\}$.

Partition Thm:

$$\mathbb{P}(W \cap N \geq n) = \mathbb{P}(W \cap N=n) + \mathbb{P}(W \cap N \geq n+1)$$

$$\Rightarrow \alpha \mathbb{P}(N \geq n) = \mathbb{P}(W \mid N=n)\mathbb{P}(N=n) + \alpha \mathbb{P}(N \geq n+1) \quad \text{if } n \geq 1$$

$$\Rightarrow \alpha \underbrace{\{\mathbb{P}(N \geq n) - \mathbb{P}(N \geq n+1)\}}_{=\mathbb{P}(N=n)} = \mathbb{P}(W \mid N=n)\mathbb{P}(N=n)$$

$$\Rightarrow \alpha \mathbb{P}(N=n) = \mathbb{P}(W \mid N=n)\mathbb{P}(N=n)$$

$$\Rightarrow \alpha = \mathbb{P}(W \mid N=n)$$ because $\mathbb{P}(N=n) > 0$ for all n.

Valid for all $n \geq 1$.

b) Suppose $n \geq 1$: Then conditioning on $\{N \geq n\}$ restricts the sample space $\Omega = \{$all paths start $\to$ finish$\}$ to only those paths with at least $n \geq 1$ visits to state A.

So we can restrict interest to a new sample space which starts at state A.

That is, $\Omega_{\{N \geq n\}} = \Omega_{\{\text{start at state } A\}}$ for all $n \geq 1$.

The same sample space is implied for all $n \geq 1$ because only the n^{th} visit to state A is relevant.

So $\mathbb{P}(W \mid N \geq n) = \mathbb{P}(W \mid \text{start at state } A)$ for any $n \geq 1$.

$\Rightarrow \mathbb{P}(W \mid N \geq n) = \text{constant} = \alpha = p_A$ in notation given.

6c) $\mathbb{E}(N \mid W) = \sum_{n=0}^{\infty} n \, \mathbb{P}(N=n \mid W)$ defn.

$= \sum_{n=1}^{\infty} n \, \mathbb{P}(N=n \mid W)$ because term in $n=0$ is 0

$= \sum_{n=1}^{\infty} n \, \frac{\mathbb{P}(W \mid N=n)\,\mathbb{P}(N=n)}{\mathbb{P}(W)}$

$= \sum_{n=1}^{\infty} n \, \frac{\alpha \, \mathbb{P}(N=n)}{\mathbb{P}(W)}$ because $\mathbb{P}(W \mid N=n) = \alpha = p_A$ for all $n = 1, 2, \ldots$ by (a) and (b)

$= \frac{p_A}{\mathbb{P}(W)} \sum_{n=1}^{\infty} n \, \mathbb{P}(N=n)$

$= \frac{p_A \, \mathbb{E}N}{\mathbb{P}(W)}$ where $\mathbb{E}N$ and $\mathbb{P}(W)$ refer to the sample space $\Omega = \{\text{start} \to \text{finish}$
$\Rightarrow \mathbb{E}N = m_S$
$\mathbb{P}(W) = p_S$

$\therefore \; \mathbb{E}(N \mid W) = \frac{p_A \, m_S}{p_S}$

Calculate m_S, p_A and p_S by first-step analysis (one each for m_S, and for p_A & p_S together).