Chapter 4: Mathematical Induction

So far in this course, we have seen some techniques for dealing with stochastic processes: first-step analysis for hitting probabilities (Chapter 2), and first-step analysis for expected reaching times (Chapter 3). We now look at another tool that is often useful for exploring properties of stochastic processes: *proof by mathematical induction.*

4.1 Proving things in mathematics

There are many different ways of constructing a formal proof in mathematics. Some examples are:

- **Proof by counterexample:** a proposition is proved to be *not generally true* because a *particular example* is found for which it is not true.
- **Proof by contradiction:** this can be used either to prove a proposition is true or to prove that it is false. To prove that the proposition is *true* (say), we start by *assuming that it is false.* We then explore the consequences of this assumption until we reach a contradiction, e.g. $0 = 1$. Therefore something must have gone wrong, and the only thing we weren't sure about was our initial assumption that the proposition is false — so our initial assumption must be wrong and the proposition is proved true.

  A famous proof of this sort is the proof that there are infinitely many prime numbers. We start by assuming that there are *finitely* many primes, so they can be listed as $p_1, p_2, \ldots, p_n$, where p_n is the largest prime number. But then the number $p_1 \times p_2 \times \ldots \times p_n + 1$ must also be prime, because it is not divisible by any of the smaller primes. Furthermore this number is definitely bigger than p_n. So we have contradicted the idea that there was a 'biggest' prime called p_n, and therefore there are infinitely many primes.
- **Proof by mathematical induction:** in mathematical induction, we start with a formula that we *suspect* is true. For example, I might *suspect* from

observation that $\sum_{k=1}^{n} k = n(n+1)/2$. I might have tested this formula for many different values of n, but of course I can never test it for *all* values of n. Therefore I need to prove that the formula is *always* true.

The idea of mathematical induction is to say: *suppose* the formula is true for all n up to the value $n = 10$ (say). Can I prove that, *if* it is true for $n = 10$, *then* it will also be true for $n = 11$? And *if* it is true for $n = 11$, then it will also be true for $n = 12$? And so on.

In practice, we usually start lower than $n = 10$. We usually take the very easiest case, $n = 1$, and prove that the formula is true for $n = 1$: LHS $= \sum_{k=1}^{1} k = 1 = 1 \times 2/2 =$ RHS. Then we prove that, *if* the formula is ever true for $n = x$, *then* it will always be true for $n = x + 1$. Because it is true for $n = 1$, it must be true for $n = 2$; and because it is true for $n = 2$, it must be true for $n = 3$; and so on, for all possible n. Thus the formula is proved.

Mathematical induction is therefore a bit like a *first-step analysis for proving things: prove that wherever we are now, the* <u>**next step**</u> *will always be OK. Then if we were OK at the very beginning, we will be OK for ever.*

The method of mathematical induction for proving results is very important in the study of Stochastic Processes. This is because a stochastic process builds up one step at a time, and mathematical induction works on the same principle.

Example: We have already seen examples of inductive-type reasoning in this course. For example, in Chapter 2 for the Gambler's Ruin problem, using the method of repeated substitution to solve for $p_x = \mathbb{P}(\text{Ruin} \mid \text{start with \$x})$, we discovered that:

- $p_2 = 2p_1 - 1$
- $p_3 = 3p_1 - 2$
- $p_4 = 4p_1 - 3$

We deduced that $p_x = xp_1 - (x - 1)$ *in general.*

To prove this properly, we should have used the method of mathematical induction.

4.2 Mathematical Induction by example

This example explains the style and steps needed for a proof by induction.

Question: Prove by induction that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for any integer n. $(\star)$

Approach: follow the steps below.

(i) First verify that the formula is true for a *base case*: usually the smallest appropriate value of n (e.g. $n = 0$ or $n = 1$). Here, the smallest possible value of n is $n = 1$, because we can't have $\sum_{k=1}^{0}$.

First verify $(\star)$ *is true when* $n = 1$*.*

$$LHS = \sum_{k=1}^{1} k = 1.$$

$$RHS = \frac{1 \times 2}{2} = 1 = LHS.$$

So $(\star)$ *is proved for* $n = 1$*.*

(ii) Next suppose that formula $(\star)$ is true for *all values of* n *up to and including some value* x. (We have already established that this is the case for $x = 1$).

Using the hypothesis that $(\star)$ is true for all values of n up to and including x, *prove* that it is therefore true for the value $n = x + 1$.

Now suppose that $(\star)$ *is true for* $n = 1, 2, \ldots, x$ *for some* x*.*

Thus we can assume that $\sum_{k=1}^{x} k = \frac{x(x+1)}{2}$*.* (a)

((a) for 'allowed' info)

We need to show that if $(\star)$ *holds for* $n = x$*, then it must also hold for* $n = x + 1$*.*

Require to prove that

$$\sum_{k=1}^{x+1} k = \frac{(x+1)(x+2)}{2} \qquad (\star\star)$$

(Obtained by putting $n = x+1$ *in* $(\star)$*).*

$$\begin{aligned}
\text{LHS of } (\star\star) = \sum_{k=1}^{x+1} k &= \sum_{k=1}^{x} k \;+\; (x+1) && \textit{by expanding the sum} \\
&= \frac{x(x+1)}{2} \;+\; (x+1) && \textit{using allowed info (a)} \\
&= (x+1)\left(\frac{x}{2}+1\right) && \textit{rearranging} \\
&= \frac{(x+1)(x+2)}{2} \\
&= \text{RHS of } (\star\star).
\end{aligned}$$

This shows that:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \text{when } n = x+1.$$

So, assuming $(\star)$ *is true for* $n = x$*, it is also true for* $n = x+1$*.*

(iii) Refer back to the base case: if it is true for $n = 1$, then it is true for $n = 1+1 = 2$ by (ii). If it is true for $n = 2$, it is true for $n = 2+1 = 3$ by (ii). We could go on forever. This proves that the formula $(\star)$ is true for all n.

We proved $(\star)$ *true for* $n = 1$*, thus* $(\star)$ *is true for all integers* $n \geq 1$*.* □

General procedure for proof by induction

The procedure above is quite standard. The inductive proof can be summarized like this:

Question: prove that $f(n) = g(n)$ for all integers $n \geq 1$.

Base case: $n = 1$. *Prove that* $f(1) = g(1)$ *using*

$$\begin{aligned} \textit{LHS} &= f(1) \\ &= \vdots \\ &= g(1) = \textit{RHS}. \end{aligned}$$

General case: suppose the formula is true for $n = x$*: so* $f(x) = g(x)$*. Prove that the formula is therefore true for* $n = x + 1$*:*

$$\begin{aligned} \textit{LHS} &= f(x+1) \\ &= \left\{ \begin{array}{c} \textit{some expression breaking down } f(x+1) \\ \textit{into } f(x) \textit{ and an extra term in } x+1 \end{array} \right\} \\ &= \left\{ \begin{array}{c} \textit{formula is true for } n = x, \textit{ so substitute} \\ f(x) = g(x) \textit{ in the line above} \end{array} \right\} \\ &= \{\textit{do some working}\} \\ &= g(x+1) \\ &= \textit{RHS}. \end{aligned}$$

Conclude: the formula is true for $n = 1$*, so it is true for* $n = 2$*,* $n = 3$*,* $n = 4, \ldots$

It is therefore true for all integers $n \geq 1$. □

4.3 Some harder examples of mathematical induction

Induction problems in stochastic processes are often trickier than usual. Here are some possibilities:

- Backwards induction: start with base case $n = N$ and go backwards, instead of starting at base case $n = 1$ and going forwards.
- Two-step induction, where the proof for $n = x + 1$ relies not only on the formula being true for $n = x$, but also on it being true for $n = x - 1$.

The first example below is hard probably because it is too easy. The second example is an example of a two-step induction.

Example 1: Suppose that $p_0 = 1$ and $p_x = \alpha p_{x+1}$ for all $x = 1, 2, \ldots$. Prove by mathematical induction that $p_n = 1/\alpha^n$ for $n = 0, 1, 2, \ldots$.

Wish to prove

$$p_n = \frac{1}{\alpha^n} \quad \text{for } n = 0, 1, 2, \ldots \qquad (\star)$$

Information given:

$$\begin{aligned} p_{x+1} &= \frac{1}{\alpha} p_x && (G_1) \\ p_0 &= 1 && (G_2) \end{aligned}$$

Base case: $n = 0$.

$$\textit{LHS} = p_0 = 1 \text{ by information given } (G_2).$$

$$\textit{RHS} = \frac{1}{\alpha^0} = \frac{1}{1} = 1 = \textit{LHS}.$$

Therefore $(\star)$ *is true for the base case* $n = 0$.

General case: suppose that $(\star)$ *is true for* $n = x$, *so we can assume*

$$p_x = \frac{1}{\alpha^x}. \qquad (a)$$

Wish to prove that $(\star)$ *is also true for* $n = x + 1$: *i.e.*

$$\textit{RTP} \qquad p_{x+1} = \frac{1}{\alpha^{x+1}}. \qquad (\star\star)$$

$$\begin{aligned} \textit{LHS of } (\star\star) = p_{x+1} &= \frac{1}{\alpha} \times p_x \qquad \textit{by given } (G_1) \\ &= \frac{1}{\alpha} \times \frac{1}{\alpha^x} \qquad \textit{by allowed (a)} \\ &= \frac{1}{\alpha^{x+1}} \\ &= \textit{RHS of } (\star\star). \end{aligned}$$

So if formula $(\star)$ *is true for* $n = x$*, it is true for* $n = x + 1$*. We have shown it is true for* $n = 0$*, so it is true for all* $n = 0, 1, 2, \ldots$*.* □

Example 2: Gambler's Ruin. In the Gambler's Ruin problem in Section 2.7, we have the following situation:

- $p_x = \mathbb{P}(\text{Ruin} \,|\, \text{start with \$x})$;
- We know from first-step analysis that $p_{x+1} = 2p_x - p_{x-1} \qquad (G_1)$
- We know from common sense that $p_0 = 1 \qquad (G_2)$
- By direct substitution into (G_1), we obtain:

$$\begin{aligned} p_2 &= 2p_1 - 1 \\ p_3 &= 3p_1 - 2 \end{aligned}$$

- We develop a suspicion that for all $x = 1, 2, 3, \ldots$,

$$p_x = xp_1 - (x - 1) \qquad (\star)$$

- We wish to prove $(\star)$ by mathematical induction.

For this example, *our given information, in* (G_1)*, expresses* p_{x+1} *in terms of both* p_x *and* p_{x-1}*, so we need two base cases. Use* $x = 1$ *and* $x = 2$*.*

Wish to prove $p_x = xp_1 - (x-1) \quad (\star)$.
Base case $x = 1$:

$$\textit{LHS} = p_1.$$

$$\textit{RHS} = 1 \times p_1 - 0 = p_1 = \textit{LHS}.$$

$$\therefore \textit{ formula } (\star) \textit{ is true for base case } x = 1.$$

Base case $x = 2$:

$$\textit{LHS} = p_2 = 2p_1 - 1 \quad \textit{by information given } (G_1)$$

$$\textit{RHS} = 2 \times p_1 - 1 = \textit{LHS}.$$

$$\therefore \textit{ formula } (\star) \textit{ is true for base case } x = 2.$$

General case: suppose that $(\star)$ *is true for* *all* x *up to* $x = k$.
So we are allowed:

$$\begin{array}{ccccc} (x = k) & p_k & = & kp_1 - (k-1) & (a_1) \\ (x = k-1) & p_{k-1} & = & (k-1)p_1 - (k-2) & (a_2) \end{array}$$

Wish to prove that $(\star)$ *is also true for* $x = k+1$, *i.e.*

$$\textit{RTP} \quad p_{k+1} = (k+1)p_1 - k. \qquad (\star\star)$$

$$\begin{aligned} \textit{LHS of } (\star\star) &= p_{k+1} \\ &= 2p_k - p_{k-1} \quad \textit{by given information } (G_1) \\ &= 2\Big\{ kp_1 - (k-1) \Big\} - \Big\{ (k-1)p_1 - (k-2) \Big\} \\ &\quad \textit{by allowed } (a_1) \textit{ and } (a_2) \\ &= p_1 \Big\{ 2k - (k-1) \Big\} - \Big\{ 2(k-1) - (k-2) \Big\} \\ &= (k+1)p_1 - k \\ &= \textit{RHS} \quad \textit{of } (\star\star) \end{aligned}$$

So if formula $(\star)$ *is true for* $x = k-1$ *and* $x = k$, *it is true for* $x = k+1$. *We have shown it is true for* $x = 1$ *and* $x = 2$, *so it is true for all* $x = 1, 2, 3, \ldots$. □