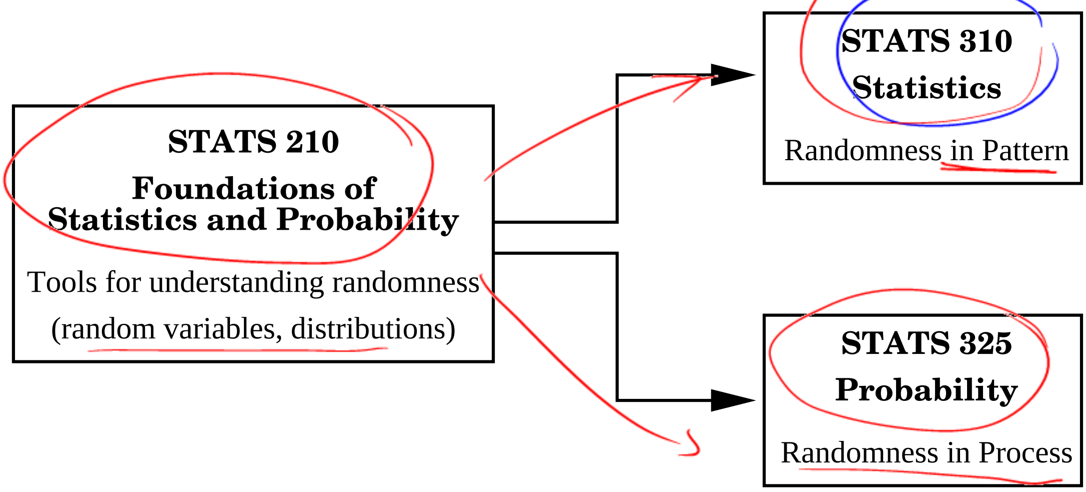
Chapter 1: Stochastic Processes

What are Stochastic Processes, and how do they fit in?



Stats 210: laid the foundations of both Statistics and Probability: the tools for understanding randomness.

Stats 310: develops the theory for understanding ***randomness in pattern:*** tools for estimating parameters (maximum likelihood), testing hypotheses, modelling patterns in data (regression models).

Stats 325: develops the theory for understanding ***randomness in process.*** A process is a sequence of events where each step follows from the last after a random choice.

What sort of problems will we cover in Stats 325?

Read

Here are some examples of the sorts of problems that we study in this course.

Gambler's Ruin

You start with \$30 and toss a fair coin repeatedly. Every time you throw a Head, you win \$5. Every time you throw a Tail, you lose \$5. You will stop when you reach \$100 or when you lose everything. What is the probability that you lose everything? ***Answer:*** 70%.

$X_1, X_2, X_3, X_4, \ldots\ldots$ IID

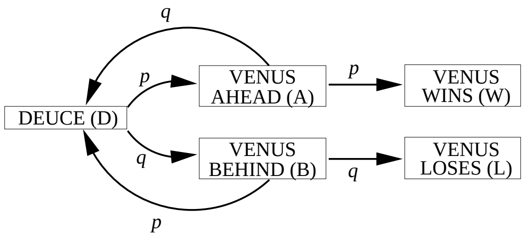
pattern

1st order process

Markov Chain

Winning at tennis

What is your probability of winning a game of tennis, starting from the even score Deuce (40-40), if your probability of winning each point is 0.3 and your opponent's is 0.7?



Answer: 15%.

Winning a lottery

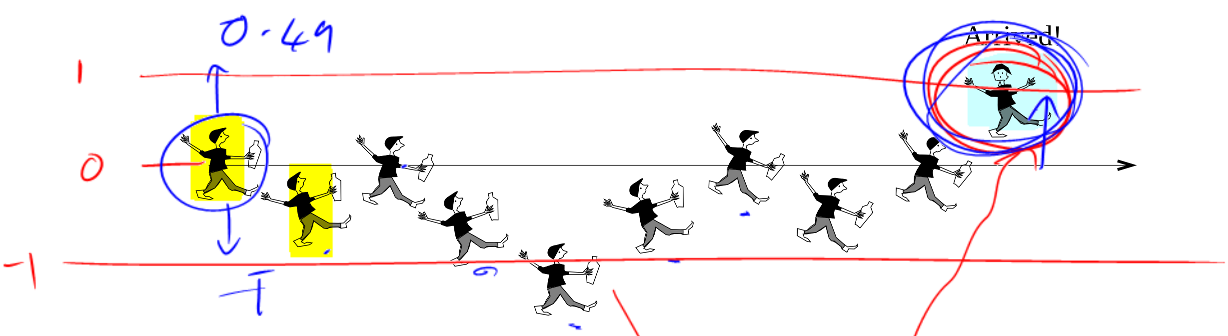
A million people have bought tickets for the weekly lottery draw. Each person has a probability of one-in-a-million of selecting the winning numbers. If more than one person selects the winning numbers, the winner will be chosen at random from all those with matching numbers.

You watch the lottery draw on TV and your numbers match the winning numbers!!! Only a one-in-a-million chance, and there were only a million players, so surely you will win the prize?

Not quite... What is the probability you will win? ***Answer:*** only 63%.

Drunkard's walk

A very drunk person staggers to left and right as he walks along. With each step he takes, he staggers one pace to the left with probability 0.5, and one pace to the right with probability 0.5. What is the expected number of paces he must take before he ends up one pace to the left of his starting point?



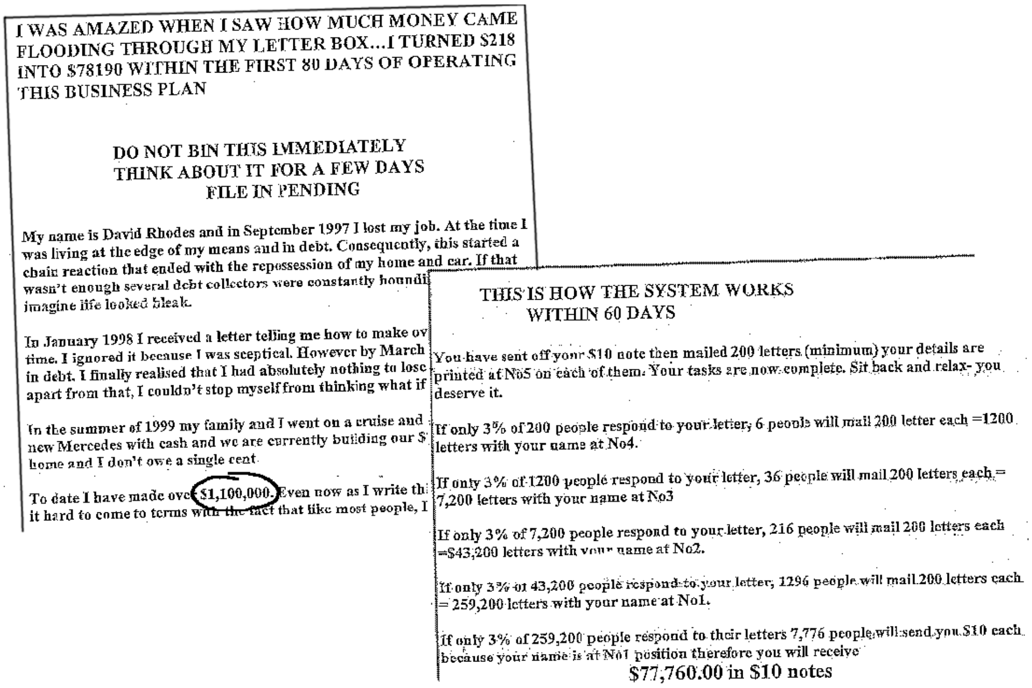
Answer: the expectation is infinite!

State space is $S = \{\dots, -2, -1, 0, 1, 2, \dots\}$

or $S = \mathbb{Z}$

Pyramid selling schemes

Have you received a chain letter like this one? Just send \$10 to the person whose name comes at the top of the list, and add your own name to the bottom of the list. Send the letter to as many people as you can. Within a few months, the letter promises, you will have received \$77,000 in \$10 notes! Will you?



I WAS AMAZED WHEN I SAW HOW MUCH MONEY CAME FLOODING THROUGH MY LETTER BOX...I TURNED $218 INTO $78190 WITHIN THE FIRST 80 DAYS OF OPERATING THIS BUSINESS PLAN

DO NOT BIN THIS IMMEDIATELY
THINK ABOUT IT FOR A FEW DAYS
FILE IN PENDING

My name is David Rhodes and in September 1997 I lost my job. At the time I was living at the edge of my means and in debt. Consequently, this started a chain reaction that ended with the repossession of my home and car. If that wasn't enough several debt collectors were constantly houndi imagine life looked bleak.

In January 1998 I received a letter telling me how to make ov time. I ignored it because I was sceptical. However by March in debt. I finally realised that I had absolutely nothing to lose apart from that, I couldn't stop myself from thinking what if

In the summer of 1999 my family and I went on a cruise and new Mercedes with cash and we are currently building our $ home and I don't owe a single cent.

To date I have made ove $1,100,000. Even now as I write th it hard to come to terms with the fact that like most people, I

THIS IS HOW THE SYSTEM WORKS
WITHIN 60 DAYS

You have sent off your $10 note then mailed 200 letters (minimum) your details are printed at No5 on each of them. Your tasks are now complete. Sit back and relax- you deserve it.

If only 3% of 200 people respond to your letter, 6 people will mail 200 letter each =1200 letters with your name at No4.

If only 3% of 1200 people respond to your letter, 36 people will mail 200 letters each = 7,200 letters with your name at No3

If only 3% of 7,200 people respond to your letter, 216 people will mail 200 letters each =$43,200 letters with your name at No2.

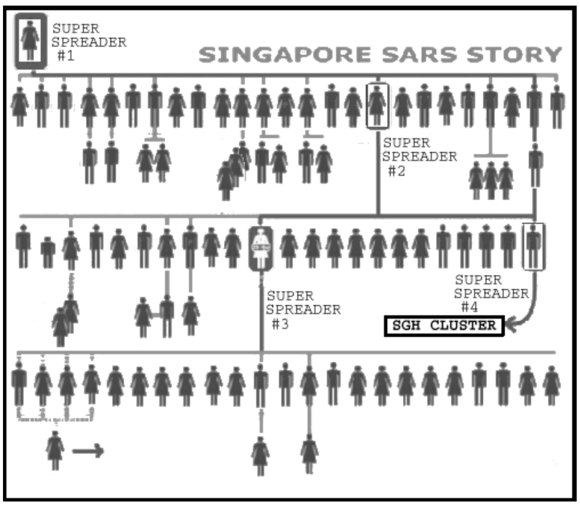
If only 3% of 43,200 people respond to your letter, 1296 people will mail 200 letters each = 259,200 letters with your name at No1.

If only 3% of 259,200 people respond to their letters 7,776 people will send you $10 each because your name is at No1 position therefore you will receive
$77,760.00 in $10 notes

Answer: it depends upon the response rate. However, with a fairly realistic assumption about response rate, we can calculate an expected return of \$76 with a 64% chance of getting nothing!

Note: Pyramid selling schemes like this are prohibited under the Fair Trading Act, and it is illegal to participate in them.

Spread of SARS



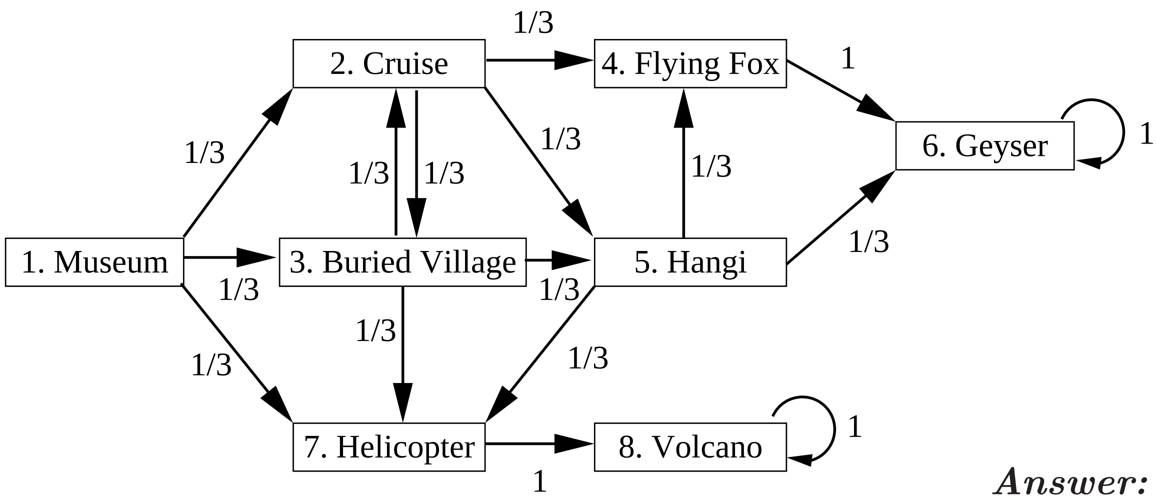
The figure to the right shows the spread of the disease SARS (Severe Acute Respiratory Syndrome) through Singapore in 2003. With this pattern of infections, what is the probability that the disease eventually dies out of its own accord?

Answer: 0.997.

Markov's Marvellous Mystery Tours

Mr Markov's Marvellous Mystery Tours promises an All-Stochastic Tourist Experience for the town of Rotorua. Mr Markov has eight tourist attractions, to which he will take his clients completely at random with the probabilities shown below. He promises at least three exciting attractions per tour, ending at either the Lady Knox Geyser or the Tarawera Volcano. (Unfortunately he makes no mention of how the hapless tourist might get home from these places.)

What is the expected number of activities for a tour starting from the museum?



Answer: 4.2.

Structure of the course

- ***Probability.*** Probability and random variables, with special focus on conditional probability. Finding hitting probabilities for stochastic processes.
- ***Expectation.*** Expectation and variance. Introduction to conditional expectation, and its application in finding expected reaching times in stochastic processes.
- ***Generating functions.*** Introduction to probability generating functions, and their applications to stochastic processes, especially the Random Walk.
- ***Branching process.*** This process is a simple model for reproduction. Examples are the pyramid selling scheme and the spread of SARS above.

- ***Markov chains.*** Almost all the examples we look at throughout the course can be formulated as Markov chains. By developing a single unifying theory, we can easily tackle complex problems with many states and transitions like Markov's Marvellous Mystery Tours above.

The rest of this chapter covers:

- quick revision of sample spaces and random variables;
- formal definition of stochastic processes.

1.1 Revision: Sample spaces and random variables

Definition: A **random experiment** is a physical situation whose outcome cannot be predicted until it is observed.

Definition: A **sample space**, Ω, is a set of possible outcomes of a random experiment.

Example:

Random experiment: Toss a coin once.
Sample space: $\Omega = \{\text{head}, \text{tail}\}$

1 0

Definition: A **random variable**, X, is defined as a function from the sample space to the real numbers: $X : \Omega \to \mathbb{R}$.

That is, *a random variable assigns a real number to every possible outcome of a random experiment.*

Example:

Random experiment: Toss a coin once.
Sample space: $\Omega = \{\text{head}, \text{tail}\}$.
An example of a random variable: $X : \Omega \to \mathbb{R}$ maps "head" $\to 1$, "tail" $\to 0$.

Essential point: $\boxed{\text{A random variable is a way of producing random real numbers.}}$

$X_1 \quad X_2 \quad X_3$

$X(1) \quad X(2) \quad X(3)$

1.2 Stochastic Processes

e.g. $\{X(0), X(1), X(2), \ldots\}$

Definition: A **stochastic process** is a family of random variables, $\{X(t) : t \in T\}$, where t usually denotes time. That is, at every time t in the set T, a random number $X(t)$ is observed.

Definition: $\{X(t) : t \in T\}$ is a **discrete-time process** if the set T is finite or countable.

In practice, this generally means $T = \{0, 1, 2, 3, \ldots\}$

Thus a discrete-time process is $\{X(0), X(1), X(2), \ldots\}$: a random number associated with every time $t = 0, 1, 2, \ldots$.

Definition: $\{X(t) : t \in T\}$ is a **continuous-time process** if *T is not finite or countable.*

In practice, this generally means $T = [0, \infty)$ or $T = [0, K]$ for some K.

Thus a continuous-time process $\{X(t) : t \in T\}$ has a random number $X(t)$ associated with EVERY INSTANT in time.

(Note that $X(t)$ need not *change* at every instant in time, but it is *allowed* to change at any time; i.e. not just at $t = 0, 1, 2, \ldots$, like a discrete-time process.)

Definition: The **state space**, S, is the set of real values that any $X(t)$ can take.

Every $X(t)$ takes a value in $\mathbb{R}$, but S will often be a smaller set: $S \subseteq \mathbb{R}$. For example, if $X(t)$ is the outcome of a coin tossed at time t, then the state space is $S = \{0, 1\}$.

Definition: The state space S is **discrete** if it is *finite or countable.* Otherwise it is *continuous.*

> The state space S is the set of **states** that the stochastic process can be in.

For Reference: Discrete Random Variables

1. Binomial distribution

Notation: $X \sim \text{Binomial}(n, p)$.

Description: number of successes in n independent trials, each with probability p of success.

Probability function:

$$f_X(x) = \mathbb{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for } x = 0, 1, \ldots, n.$$

Mean: $\mathbb{E}(X) = np$.

Variance: $\text{Var}(X) = np(1-p) = npq$, where $q = 1 - p$.

Sum: If $X \sim \text{Binomial}(n, p)$, $Y \sim \text{Binomial}(m, p)$, and X and Y are **independent**, then

$$X + Y \sim \text{Bin}(n + m,\ p).$$

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2. Poisson distribution

Notation: $X \sim \text{Poisson}(\lambda)$.

Description: arises out of the Poisson process as the number of events in a fixed time or space, when events occur at a constant average rate. Also used in many other situations.

Probability function: $f_X(x) = \mathbb{P}(X = x) = \dfrac{\lambda^x}{x!} e^{-\lambda}$ for $x = 0, 1, 2, \ldots$

Mean: $\mathbb{E}(X) = \lambda$.

Variance: $\text{Var}(X) = \lambda$.

Sum: If $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$, and X and Y are **independent**, then

$$X + Y \sim \text{Poisson}(\lambda + \mu).$$

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3. Geometric distribution

Notation: $X \sim \text{Geometric}(p)$.

Description: number of failures before the **first** success in a sequence of independent trials, each with $\mathbb{P}(\text{success}) = p$.

Probability function: $f_X(x) = \mathbb{P}(X = x) = (1-p)^x p \quad \text{for} \quad x = 0, 1, 2, \ldots$

Mean: $\mathbb{E}(X) = \dfrac{1-p}{p} = \dfrac{q}{p}$, where $q = 1 - p$.

Variance: $\text{Var}(X) = \dfrac{1-p}{p^2} = \dfrac{q}{p^2}$, where $q = 1 - p$.

Sum: if $X_1, \ldots, X_k$ are **independent**, and each $X_i \sim \text{Geometric}(p)$, then

$$X_1 + \ldots + X_k \sim \text{Negative Binomial}(k, p).$$

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4. Negative Binomial distribution

Notation: $X \sim \text{NegBin}(k, p)$.

Description: number of failures before the ***k*th** success in a sequence of independent trials, each with $\mathbb{P}(\text{success}) = p$.

Probability function:

$$f_X(x) = \mathbb{P}(X = x) = \binom{k+x-1}{x} p^k (1-p)^x \quad \text{for} \quad x = 0, 1, 2, \ldots$$

Mean: $\mathbb{E}(X) = \dfrac{k(1-p)}{p} = \dfrac{kq}{p}$, where $q = 1 - p$.

Variance: $\text{Var}(X) = \dfrac{k(1-p)}{p^2} = \dfrac{kq}{p^2}$, where $q = 1 - p$.

Sum: If $X \sim \text{NegBin}(k,\ p)$, $Y \sim \text{NegBin}(m,\ p)$, and X and Y are **independent**, then

$$X + Y \sim \text{NegBin}(k + m,\ p).$$

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5. Hypergeometric distribution

Notation: $X \sim \text{Hypergeometric}(N, M, n)$.

Description: Sampling without replacement from a finite population. Given N objects, of which M are 'special'. Draw n objects without replacement. X is the number of the n objects that are 'special'.

Probability function:

$$f_X(x) = \mathbb{P}(X = x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \quad \text{for} \quad \begin{cases} x = \max(0,\ n + M - N) \\ \text{to}\ \ x = \min(n,\ M). \end{cases}$$

Mean: $\mathbb{E}(X) = np$, where $p = \dfrac{M}{N}$.

Variance: $\text{Var}(X) = np(1-p)\left(\dfrac{N-n}{N-1}\right)$, where $p = \dfrac{M}{N}$.

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6. Multinomial distribution

Notation: $\boldsymbol{X} = (X_1, \ldots, X_k) \sim \text{Multinomial}(n;\ p_1, p_2, \ldots, p_k)$.

Description: there are n independent trials, each with k possible outcomes. Let $p_i = \mathbb{P}(\text{outcome } i)$ for $i = 1, \ldots k$. Then $\boldsymbol{X} = (X_1, \ldots, X_k)$, where X_i is the number of trials with outcome i, for $i = 1, \ldots, k$.

Probability function:

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \mathbb{P}(X_1 = x_1, \ldots, X_k = x_k) = \frac{n!}{x_1! \ldots x_k!} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k}$$

$$\text{for } x_i \in \{0, \ldots, n\}\ \forall_i \text{ with } \sum_{i=1}^{k} x_i = n, \text{ and where } p_i \geq 0\ \forall_i,\ \sum_{i=1}^{k} p_i = 1.$$

Marginal distributions: $X_i \sim \text{Binomial}(n, p_i)$ for $i = 1, \ldots, k$.

Mean: $\mathbb{E}(X_i) = np_i$ for $i = 1, \ldots, k$.

Variance: $\text{Var}(X_i) = np_i(1 - p_i)$, for $i = 1, \ldots, k$.

Covariance: $\text{cov}(X_i, X_j) = -np_ip_j$, for all $i \neq j$.

Continuous Random Variables

1. Uniform distribution

Notation: $X \sim \text{Uniform}(a, b)$.

Probability density function (pdf): $f_X(x) = \dfrac{1}{b-a}$ for $a < x < b$.

Cumulative distribution function:

$$F_X(x) = \mathbb{P}(X \leq x) = \frac{x-a}{b-a} \quad \text{for } a < x < b.$$

$$F_X(x) = 0 \text{ for } x \leq a, \text{ and } F_X(x) = 1 \text{ for } x \geq b.$$

Mean: $\mathbb{E}(X) = \dfrac{a+b}{2}$.

Variance: $\text{Var}(X) = \dfrac{(b-a)^2}{12}$.

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2. Exponential distribution

Notation: $X \sim \text{Exponential}(\lambda)$.

Probability density function (pdf): $f_X(x) = \lambda e^{-\lambda x}$ for $0 < x < \infty$.

Cumulative distribution function:

$$F_X(x) = \mathbb{P}(X \leq x) = 1 - e^{-\lambda x} \quad \text{for } 0 < x < \infty.$$

$$F_X(x) = 0 \text{ for } x \leq 0.$$

Mean: $\mathbb{E}(X) = \dfrac{1}{\lambda}$.

Variance: $\text{Var}(X) = \dfrac{1}{\lambda^2}$.

Sum: if $X_1, \ldots, X_k$ are **independent**, and each $X_i \sim \text{Exponential}(\lambda)$, then

$$X_1 + \ldots + X_k \sim \text{Gamma}(k, \lambda).$$

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3. Gamma distribution

Notation: $X \sim \text{Gamma}(k,\ \lambda)$.

Probability density function (pdf):

$$f_X(x) = \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x} \quad \text{for } 0 < x < \infty,$$

where $\Gamma(k) = \int_0^\infty y^{k-1} e^{-y}\, dy$ (the Gamma function).

Cumulative distribution function: no closed form.

Mean: $\mathbb{E}(X) = \dfrac{k}{\lambda}$.

Variance: $\text{Var}(X) = \dfrac{k}{\lambda^2}$.

Sum: if $X_1, \ldots, X_n$ are **independent**, and $X_i \sim \text{Gamma}(k_i,\ \lambda)$, then

$$X_1 + \ldots + X_n \sim \text{Gamma}(k_1 + \ldots + k_n,\ \lambda).$$

---

4. Normal distribution

Notation: $X \sim \text{Normal}(\mu,\ \sigma^2)$.

Probability density function (pdf):

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\{-(x-\mu)^2/2\sigma^2\}} \quad \text{for } -\infty < x < \infty.$$

Cumulative distribution function: no closed form.

Mean: $\mathbb{E}(X) = \mu$.

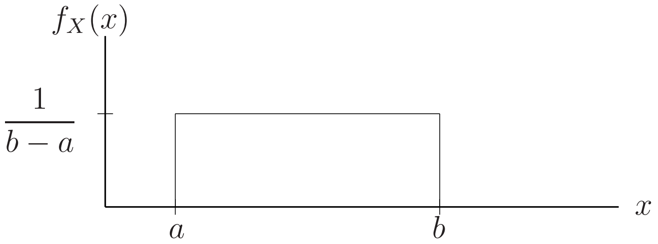
Variance: $\text{Var}(X) = \sigma^2$.

Sum: if $X_1, \ldots, X_n$ are **independent**, and $X_i \sim \text{Normal}(\mu_i,\ \sigma_i^2)$, then

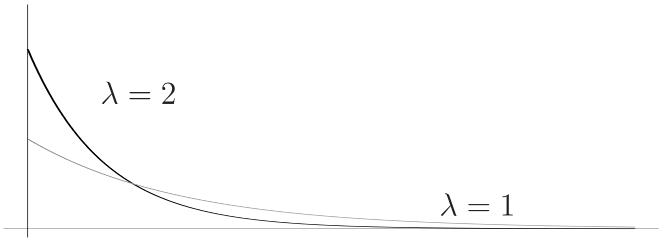
$$X_1 + \ldots + X_n \sim \text{Normal}(\mu_1 + \ldots + \mu_n,\ \sigma_1^2 + \ldots + \sigma_n^2).$$

Probability Density Functions

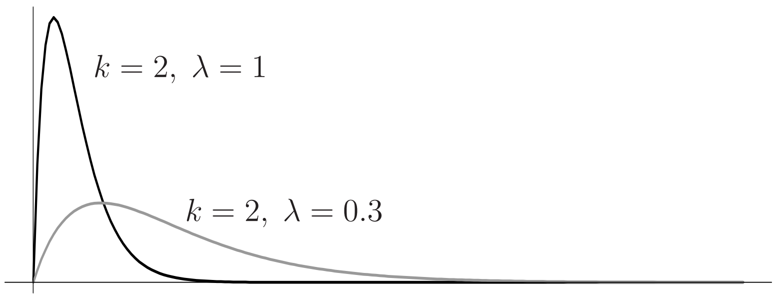
Uniform(a, b)



Exponential(λ)



Gamma(k, λ)



Normal(μ, σ^2)