Bonus Qs : Answers will not be generally released (yet)
— ask if you want them.

Rvs have: $\begin{cases} \text{probability function, } f_x(x) = P(X=x) \\ \text{PDF if continuous, } f_x(x). \end{cases}$

also   Cumulative distn function: $F_x(x) = P(X \le x)$.

Chapter 4: Generating Functions

This chapter looks at Probability Generating Functions (PGFs) for **discrete** random variables. PGFs are useful tools for dealing with sums and limits of random variables. For some stochastic processes, they also have a special role in telling us whether a process will *ever* reach a particular state.

$\leftarrow$ infinite

$X + Y$ pain

$E(s^{X+Y})$ product

$\searrow E(s^X) E(s^Y)$
if X, Y indept.

By the end of this chapter, you should be able to:

- find the sum of Geometric, Binomial, and Exponential series;
- know the definition of the PGF, and use it to calculate the mean, variance, and probabilities;
- calculate the PGF for Geometric, Binomial, and Poisson distributions;
- calculate the PGF for a randomly stopped sum;
- calculate the PGF for first reaching times in the random walk;
- use the PGF to determine whether a process will *ever* reach a given state.

4.1 Common sums

1. Geometric Series

$$1 + r + r^2 + r^3 + \cdots = \sum_{x=0}^{\infty} r^x = \frac{1}{1-r} \qquad \text{when } |r| < 1.$$

Proof: let $S = \sum_{x=0}^{\infty} r^x = 1 + r + r^2 + \cdots$

then $rS = r + r^2 + r^3 + \cdots$

So $S - rS = 1 \Rightarrow S(1-r) = 1, \ S = \frac{1}{1-r}$

This formula proves that $\sum_{x=0}^{\infty} P(X = x) = 1$ when $X \sim \text{Geometric}(p)$:

LEARN AND RECOGNISE

If $X \sim$ Geometric (p)

then $P(X = x) = p(1-p)^x$ for $x = 0, 1, \ldots$

So $\sum_{x=0}^{\infty} P(X=x) = \sum_{x=0}^{\infty} p(1-p)^x$

$= p \sum_{x=0}^{\infty} (1-p)^x$ let $r = 1-p$ in formula above.

$= p * \dfrac{1}{1 - \{1-p\}}$ $\leftarrow$ because $|1-p| < 1$

$= \dfrac{p}{p}$ if p is a probability > 0.

$= 1$

Prob Generating Function, PGF, $G_x(s)$

$$f_x(0) = \mathbb{P}(X=0) = P_0$$
$$f_x(1) = \mathbb{P}(X=1) = P_1$$
$$f_x(2) = \mathbb{P}(X=2) = P_2$$

$f_x(x)$

$F_x(x)$

where $s \in \mathbb{R}$

ie. $G_x(s)$ is a function from $\mathbb{R} \to \mathbb{R}$

take any real number s, apply $G_x(s)$ to it, we get ano real number.

$x =$ value of the dist (value X can take)

Power Series

p's are just constant coefficients.

$$G(s) = P_0 + P_1 s + P_2 s^2 + P_3 s^3 + P_4 s^4 + \cdots$$

$s \in \mathbb{R}$

If $s = 0$, $\quad G(s) = G(0) = P_0$

$$G'(s) = P_1 + 2P_2 s + 3P_3 s^2 + 4P_4 s^4 + \cdots$$

If $s = 0$, $\quad G'(s) = G'(0) = P_1$

If $s = 0$, $\quad G''(s) = G''(0) = 2P_2 \quad \Rightarrow$ get P_2.

Idea is that $G(s)$ enables us to recover ALL coefficients $P_0, P_1, P_2, \cdots$ eventually.

So $G(s)$ contains all info there is to know about these $P_0, P_1, P_2 \cdots$

$$G_x(s) = \mathbb{P}(X=0)s^0 + \mathbb{P}(X=1)s^1 + \mathbb{P}(X=2)s^2 + \mathbb{P}(X=3)s^3 + \cdots$$

$$= \sum_{x=0}^{\infty} \mathbb{P}(X=x) s^x$$

like $\mathbb{E}(g(X))$

where $g(X) = s^x$

$$= \mathbb{E}(s^X)$$

2. Binomial Theorem

For any $p, q \in \mathbb{R}$, and integer n,

$$(p+q)^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}.$$

Note: $\binom{n}{x} = {}^nC_x = \frac{n!}{(n-x)!\,x!}$, $\quad$ ${}^nC_r$ button on calculator.

The Binomial Theorem proves that $\sum_{x=0}^{n} \mathbb{P}(X = x) = 1$ when $X \sim \text{Binomial}(n, p)$:

$$\sum_{x=0}^{n} \mathbb{P}(X=x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for } X \sim \text{Bin}(n,p)$$

$$= \{ p + (1-p) \}^n \quad \text{using Binomial Thm with } q = 1-p$$

$$= 1^n$$

$$= 1.$$

3. Exponential Power Series

For any $\lambda \in \mathbb{R}$, $\quad \sum_{x=0}^{\infty} \dfrac{\lambda^x}{x!} = e^{\lambda}$.

This proves that $\sum_{x=0}^{\infty} \mathbb{P}(X = x) = 1$ when $X \sim \text{Poisson}(\lambda)$:

$$\mathbb{P}(X=x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad \text{for} \quad x = 0, 1, 2, \dots \text{ if } X \sim \text{Poisson}(\lambda),$$

So $\displaystyle\sum_{x=0}^{\infty} \mathbb{P}(X=x) = \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!}$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda} * e^{+\lambda} = e^{-\lambda+\lambda} = e^{0} = 1.$$

Note: Another useful identity is:

$$e^{\lambda} = \lim_{n \to \infty} \left\{ 1 + \frac{\lambda}{n} \right\}^n \quad \text{for } \lambda \in \mathbb{R}. \quad \left(\begin{array}{c} \text{don't} \\ \text{learn} \end{array} \right)$$

4.2 Probability Generating Functions

The **probability generating function (PGF)** is a useful tool for dealing with **_discrete_** random variables taking values $0, 1, 2, \ldots$. Its particular strength is that it gives us an easy way of characterizing the distribution of $X + Y$ when X and Y are independent. In general it is difficult to find the distribution of a sum using the traditional probability function. The PGF transforms a sum into a product and enables it to be handled much more easily.

Sums of random variables are particularly important in the study of stochastic processes, because many stochastic processes are formed from the sum of a sequence of repeating steps: for example, the Gambler's Ruin from Section 2.7.

The name _probability generating function_ also gives us another clue to the role of the PGF. The PGF can be used to generate all the probabilities of the distribution. This is generally tedious and is not often an efficient way of calculating probabilities. However, the fact that it _can_ be done demonstrates that the PGF tells us everything there is to know about a distribution.

Definition: Let X be a discrete random variable taking values in the non-negative integers $\{0, 1, 2, \ldots\}$. The **probability generating function (PGF)** of X is

$$G_X(s) = \mathbb{E}(s^X), \quad \text{for all } s \in \mathbb{R} \text{ for which the expectation sum converges.}$$

Calculating the probability generating function

$$G_X(s) = \mathbb{E}(s^X) = \sum_{x=0}^{\infty} s^x \, \mathbb{P}(X = x).$$

Just $\mathbb{E}(g(X))$ where $g(X) = s^X$.

Properties of the PGF:

1. $G_X(0) = \mathbb{P}(X = 0)$:

$s = 0 \longrightarrow$ get back $p_0 = \mathbb{P}(X = 0)$.

$$G_X(0) = \mathbb{E}(0^X) = 0^0 \, \mathbb{P}(X = 0) + 0^1 \, \mathbb{P}(X = 1) + 0^2 \, \mathbb{P}(X = 2) + \ldots$$

Note $0^0 = 1$ (defined as $\lim_{s \to 0} s^0 = 1$).

So $G_X(0) = 1 * \mathbb{P}(X = 0)$.

$\downarrow \ s=1$

2. $G_X(1) = 1$: $\quad G_X(1) = \sum_{x=0}^{\infty} 1^x \, \mathbb{P}(X=x) = \sum_{x=0}^{\infty} \mathbb{P}(X=x) = 1.$

(true for any non-defective r.v. X : see later).

Example 1: Binomial Distribution

Let $X \sim \text{Binomial}(n, p)$, so $\mathbb{P}(X = x) = \binom{n}{x} p^x q^{n-x}$ for $x = 0, 1, \ldots, n$.

$G_X(s) = \mathbb{E}(s^X)$

think of it as $\mathbb{E}(g(X))$ for $g(X)$ a function of X : here $g(X) = s^X$

$\quad = \sum_{x=0}^{n} s^x \, \mathbb{P}(X=x)$

so recall $\mathbb{E}(g(X)) = \sum_{x=0}^{n} g(x) \mathbb{P}(X=x)$

n, not ∞ (1 mark) for $X \sim \text{Bin}$

$\quad = \sum_{x=0}^{n} s^x \binom{n}{x} p^x (1-p)^{n-x}$

$\quad = \sum_{x=0}^{n} \binom{n}{x} (ps)^x (1-p)^{n-x}$

$\boxed{G_X(s) = \{ps + 1-p\}^n \quad \text{by the Binomial Theorem.}}$ | Valid for all $s \in \mathbb{R}$

$(\text{if } q = 1-p) \quad = \{ps + q\}^n$

$G_X(s) = \mathbb{E}(s^X)$ for $s \in \mathbb{R}$

X ~ Bin(n=4, p=0.2)

$G_X(s) = (0.2s + 0.8)^4$

Check $G_X(0)$:

Try $s = 0$:

$G_X(0) = (p * 0 + 1-p)^n$

$\quad = (1-p)^n$

$\quad = \mathbb{P}(X=0) \checkmark$

Check $G_X(1)$:

$G_X(1) = (p * 1 + 1-p)^n$

$(s=1) \quad = (p + 1 - p)^n$

$\quad = 1^n$

$\therefore G_X(1) = 1.$ as we thought it should be (see top of page).
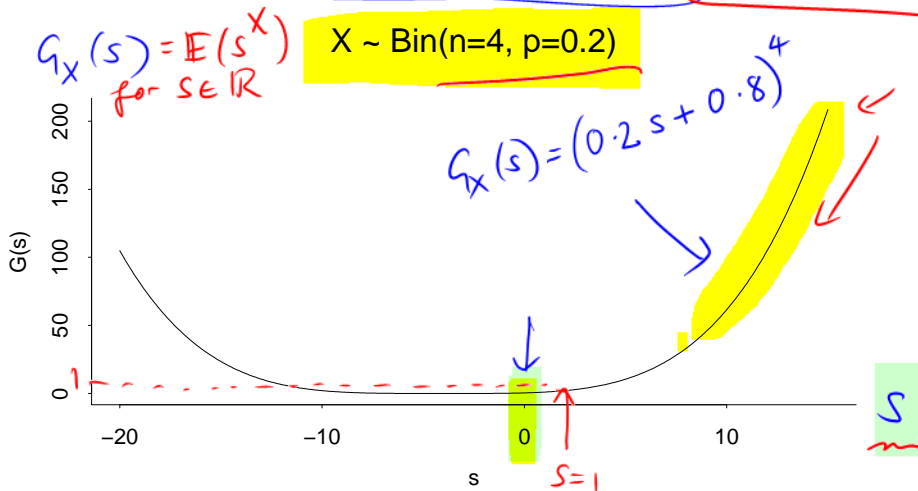
$$\boxed{G_X(s) = \mathbb{E}(s^X)}$$ $\qquad s \in \mathbb{R}$

$$= \boxed{P_0} + \boxed{P_1}s + \boxed{P_2}s^2 + \boxed{P_3}s^3 + \dots$$

where $\qquad \boxed{P_x = \mathbb{P}(X=x)}$

Example 2: __Poisson Distribution__

Let $X \sim \text{Poisson}(\lambda)$, so $\mathbb{P}(X=x) = \dfrac{\lambda^x}{x!}e^{-\lambda}$ for $x = 0,1,2,\dots$.

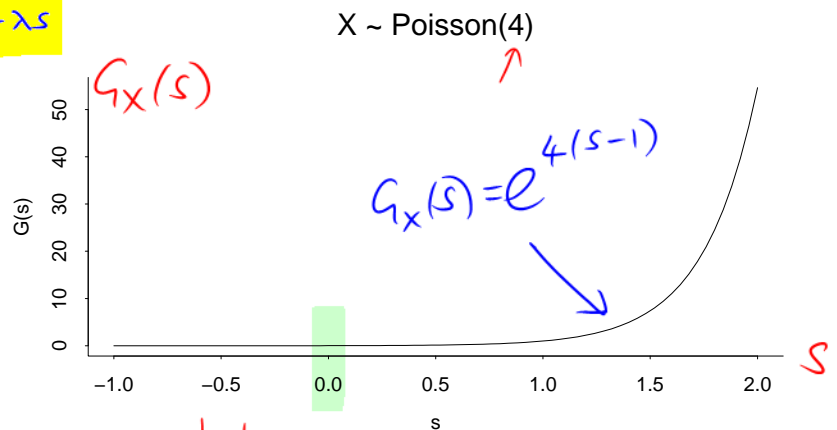$$G_X(s) = \mathbb{E}(s^X) = \sum_{x=0}^{\infty} s^x \frac{\lambda^x}{x!} e^{-\lambda}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda s)^x}{x!}$$

$$= e^{-\lambda} * e^{\lambda s} \qquad \text{(exponential series)}.$$ — true for $s \in \mathbb{R}$

$$= e^{-\lambda + \lambda s}$$

$$\Rightarrow \boxed{G_X(s) = e^{\lambda(s-1)}}$$

Valid for $s \in \mathbb{R}$.

X ~ Poisson(4)

$$G_X(s) = e^{4(s-1)}$$

below $s = -5$, there's no problem with the function $\dfrac{p}{1-qs}$ but it no longer represents the expectation $\mathbb{E}(s^X)$.

Example 3: __Geometric Distribution__

Let $X \sim \text{Geometric}(p)$, so $\boxed{\mathbb{P}(X=x) = \boxed{p(1-p)^x = pq^x}}$ for $x = 0,1,2,\dots$, where $q = 1-p$.

$X \sim \text{Geom}(p=0.8) \Rightarrow \dfrac{1}{q} = 5$

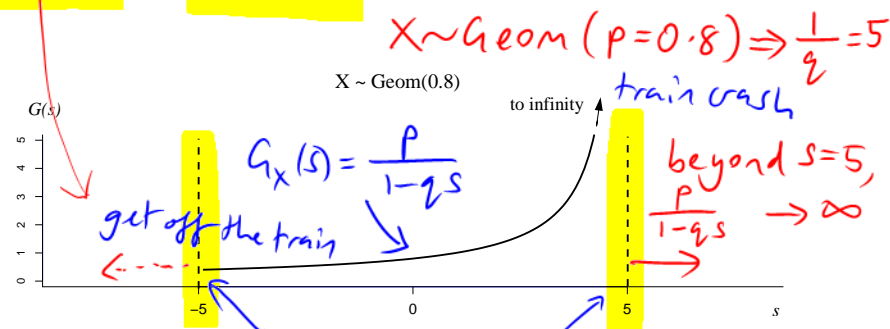X ~ Geom(0.8)

$$G_X(s) = \mathbb{E}(s^X)$$

$$= \sum_{x=0}^{\infty} s^x p q^x$$

$$= p \sum_{x=0}^{\infty} (qs)^x$$

$$= p \cdot \frac{1}{1-qs} \quad \text{for } \boxed{|qs| < 1} \quad (\text{Geometric series})$$

$$\Rightarrow \boxed{G_X(s) = \frac{p}{1-qs}} \quad \text{for } \boxed{|s| < \frac{1}{q}}.$$

$G_X(s) = \dfrac{p}{1-qs}$

get off the train

to infinity — train crash

beyond $s=5$, $\dfrac{p}{1-qs} \to \infty$

PGF for Geometric distn.

4.3   Using the probability generating function to calculate probabilities

The probability generating function gets its name because the power series can be expanded and differentiated to reveal the individual probabilities. Thus,

given only the PGF, $G_X(s) = E(s^X)$, we can recover all probabilities $P(X = x)$.

For shorthand, write $p_x = \mathbb{P}(X = x)$. Then

$$G_X(s) = E(s^X) = \sum_{x=0}^{\infty} p_x s^x = p_0 + p_1 s + p_2 s^2 + p_3 s^3 + \dots$$

Thus
$$p_0 = \mathbb{P}(X = 0) = G_X(0). \quad = \frac{1}{0!} G_X(0)$$

First derivative:
$$G_X'(s) = p_1 + 2p_2 s + 3p_3 s^2 + 4p_4 s^3 + \dots$$

Thus
$$p_1 = \mathbb{P}(X = 1) = G_X'(0). \quad = \frac{1}{1!} G_X'(0)$$

Second derivative:
$$G_X''(s) = 2p_2 + 3*2 p_3 s + 4*3 p_4 s^2 + \dots$$

Thus
$$p_2 = \mathbb{P}(X = 2) = \frac{1}{2} G_X''(0) = \frac{1}{2!} G_X''(0)$$

Third derivative:
$$G_X'''(s) = (3*2*1) p_3 + (4*3*2) p_4 s + \dots$$

Thus
$$p_3 = \mathbb{P}(X = 3) = \frac{1}{3!} G_X'''(0).$$

In general:

$$p_n = \mathbb{P}(X = n) = \frac{1}{n!} G_X^{(n)}(0) = \frac{1}{n!} \frac{d^n G_X}{ds^n} \Bigg|_{s=0}$$

Example: Let X be a discrete random variable with PGF $G_X(s) = \dfrac{s}{5}(2 + 3s^2)$.

Find the distribution of X.

$$G_X(s) = \frac{2}{5}s + \frac{3}{5}s^3 \leftarrow$$

$$\frac{1}{0!}G_X(0) = 0 \qquad = \mathbb{P}(X=0)$$

$$G_X'(s) = \frac{2}{5} + \frac{9}{5}s^2 \leftarrow$$

$$\frac{1}{1!}G_X'(0) = \frac{2}{5} \qquad = \mathbb{P}(X=1)$$

$$G_X''(s) = \frac{18}{5}s \qquad \leftarrow$$

$$\frac{1}{2!}G_X''(0) = 0 \qquad = \mathbb{P}(X=2)$$

$$G_X'''(s) = \frac{18}{5} \cdot \qquad \leftarrow$$

$$\frac{1}{3!}G_X'''(0) = \frac{1}{3!} * \frac{18}{5} = \frac{3}{5} = \mathbb{P}(X=3)$$

$$G_X^{(r)}(s) = \boxed{0} \text{ for all } r \geq 4$$

$$\frac{1}{r!}G_X^{(r)}(0) = 0 = \mathbb{P}(X=r) \text{ for all } r = 4, 5, 6, \ldots$$

Thus
$$X = \begin{cases} 1 & \text{w.p. } 2/5 \\ 3 & \text{w.p. } 3/5 \end{cases}$$

Uniqueness of the PGF

The formula $p_n = \mathbb{P}(X=n) = \left(\dfrac{1}{n!}\right)G_X^{(n)}(0)$ shows that the whole sequence of probabilities $p_0, p_1, p_2, \ldots$ is determined by the values of the PGF and its derivatives at $s = 0$. It follows that the PGF specifies a **unique** set of probabilities.

Fact: If two power series agree on any interval containing 0, however small, then all terms of the two series are equal.

Formally: let $A(s)$ and $B(s)$ be PGFs with $A(s) = \sum_{n=0}^{\infty} a_n s^n$, $B(s) = \sum_{n=0}^{\infty} b_n s^n$. If there exists some $R' > 0$ such that $A(s) = B(s)$ for all $-R' < s < R'$, then $a_n = b_n$ for all n.

Practical use: If we can show that two random variables have the same PGF in some interval containing 0, then we have shown that the two random variables must have the same distribution.

Another way of expressing this is to say that the PGF of X tells us everything there is to know about the distribution of X.

X = random variable, values $0, 1, 2, \ldots$

PGF: $G_X(s)$

$s \in \mathbb{R}$

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4.4 Expectation and moments from the PGF

As well as calculating probabilities, we can also use the PGF to calculate the moments of the distribution of X. The moments of a distribution are *the mean, variance, etc.*

Theorem 4.4: Let X be a discrete random variable with PGF $G_X(s)$. Then: ✓

on formula sheet

1. $\mathbb{E}(X) = G_X'(1)$

2. $\mathbb{E}\left\{ \underbrace{X(X-1)(X-2)\cdots(X-k+1)} \right\} = G_X^{(k)}(1)$

   (This is the k'th factorial moment of X.) $= \dfrac{d^k G_X}{ds^k}\Big|_{s=1}$.
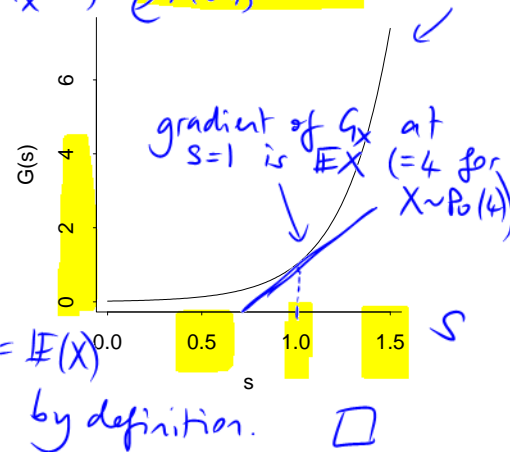
Proof: (Sketch: see Section 4.8 for more details)

1. $G_X(s) = \displaystyle\sum_{x=0}^{\infty} s^x p_x$ by definition. $G_X(s) = e^{4(s-1)}$

   $X \sim$ Poisson(4)

Differentiate:

   $G_X'(s) = \displaystyle\sum_{x=0}^{\infty} x\, s^{x-1} p_x$

$\Rightarrow$ Subst. $s=1$,

   then $G_X'(1) = \displaystyle\sum_{x=0}^{\infty} x\, 1^{x-1} p_x = \sum_{x=0}^{\infty} x\, p_x = \mathbb{E}(X)$

   by definition. $\square$

gradient of G_X at $s=1$ is $\mathbb{E}X$ ($=4$ for $X \sim Po(4)$)

2. $G_X^{(k)}(s) = \dfrac{d^k G_X(s)}{ds^k} = \displaystyle\sum_{x=k}^{\infty} x(x-1)(x-2)\ldots(x-k+1)s^{x-k}p_x$

   so $G_X^{(k)}(1) = \displaystyle\sum_{x=k}^{\infty} x(x-1)(x-2)\ldots(x-k+1)p_x$

   $= \mathbb{E}\left\{X(X-1)(X-2)\ldots(X-k+1)\right\}.$ $\square$

Prove $\mathbb{E}X = \lambda$ if $X \sim \text{Poisson}(\lambda)$

$$\mathbb{E}X = \sum_{x=0}^{\infty} x \frac{\lambda^x}{x!} e^{-\lambda} = \text{lots of work} \ldots = \lambda$$

Example: Let $X \sim \text{Poisson}(\lambda)$. The PGF of X is $G_X(s) = e^{\lambda(s-1)}$. Find $\mathbb{E}(X)$ and $\text{Var}(X)$.

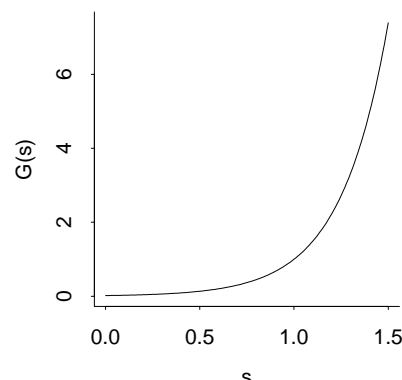X ~ Poisson(4)

Solution: $G_X(s) = e^{\lambda(s-1)}$

$$\Rightarrow \quad G_X'(s) = \lambda e^{\lambda(s-1)}$$

$$\Rightarrow \quad \mathbb{E}(X) = G_X'(1)$$
$$= \lambda e^{\lambda(1-1)}$$
$$= \lambda e^0$$

$$\Rightarrow \quad \underline{\mathbb{E}X = \lambda} \qquad \smiley$$

As we already knew, but it takes very little work.

For the variance, we need $\mathbb{E}(X^2) - (\mathbb{E}X)^2$. Note that the PGF will not give $\mathbb{E}(X^2)$ directly, just gives $\mathbb{E}[X(X-1)] = \mathbb{E}(X^2) - \mathbb{E}X$.

Consider $\mathbb{E}[X(X-1)] = G_X''(1) = \lambda^2 e^{\lambda(s-1)}\big|_{s=1} = \lambda^2$.

So $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2$
$$= \mathbb{E}[X(X-1)] + \mathbb{E}X - (\mathbb{E}X)^2$$
$$= \lambda^2 + \lambda - \lambda^2$$
$$\therefore \underline{\text{Var}(X) = \lambda}$$

we knew previously $\mathbb{E}X = \text{Var}(X) = \lambda$ when $X \sim \text{Poisson}(\lambda)$, but this proof is easy.

4.5 Probability generating function for a sum of independent r.v.s

One of the PGF's greatest strengths is that it turns a sum into a product:

$$\mathbb{E}\left(s^{(X_1+X_2)}\right) = \mathbb{E}\left(s^{X_1} s^{X_2}\right).$$

pgf of $T = X_1 + X_2$

This makes the PGF useful for finding the probabilities and moments of a SUM of INDEPENDENT random variables: because if X_1 & X_2 are indept, then $\mathbb{E}(s^{X_1} s^{X_2}) = \mathbb{E}(s^{X_1})\mathbb{E}(s^{X_2})$.

Theorem 4.5: Suppose that $X_1, \ldots, X_n$ are ***independent*** random variables, and let $Y = X_1 + \ldots + X_n$. Then

$$G_Y(s) = \prod_{i=1}^{n} G_{X_i}(s).$$

Say I have r.v.s X and Y , indept.

Want to know the distn of $T = X + Y$.

Using the probability function:

$$P(T = t) = \sum_{x=0}^{t} P(X = x \text{ AND } Y = t-x)$$

e.g.
if $t=10$

x	y
0	10
1	9
2	8
3	7
⋮	⋮
10	0

$$= \sum_{x=0}^{t} P(X = x) \, P(Y = t-x) \text{ by indep.}$$

this is called a CONVOLUTION of X & Y.

$$= \text{a lot of work} \neq \text{maybe an intractable sum.}$$

If X, Y, Z all indept, want distn of $T = X + Y + Z$,

$$P(T = t) = \sum_{x=0}^{t} \sum_{y=0}^{t-x} P(X = x) \, P(Y = y) \, P(Z = t-x-y)$$

Central Limit Theorem is exciting because we ESCAPE from convolutions via a Normal approximation:

for LARGE n, $X_1, \ldots, X_n$ indept.

  $T = X_1 + \ldots + X_n \approx$ approx Normal ☺

  CLT works only for LARGE n.
    What if n is NOT large?   ⟶   PGF ☺

Proof:

$$G_Y(s) = \mathbb{E}(s^Y)$$
$$= \mathbb{E}(s^{X_1 + \cdots + X_n})$$
$$= \mathbb{E}(s^{X_1} s^{X_2} \cdots s^{X_n}) \quad \text{always true}$$

1 mark

$$= \mathbb{E}(s^{X_1})\,\mathbb{E}(s^{X_2}) \cdots \mathbb{E}(s^{X_n}) \quad \text{because } X_1, \ldots, X_n \text{ are independent by definition.}$$
$$= G_{X_1}(s)\, G_{X_2}(s) \cdots G_{X_n}(s)$$
$$= \prod_{i=1}^n G_{X_i}(s) \quad \text{as required.} \quad \square$$

Example: Suppose that X and Y are independent with $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$. Find the distribution of $X + Y$.

Solution:
$$G_{X+Y}(s) = G_X(s)\, G_Y(s) \quad \text{by indep of } X \text{ \& } Y$$
$$= e^{\lambda(s-1)}\, e^{\mu(s-1)}$$
$$= e^{(\lambda + \mu)(s-1)}$$

But this is the PGF of a Poisson $(\lambda+\mu)$ distribution. So, by the uniqueness of PGFs, we have proved that if $X \sim$ Poisson (λ) and $Y \sim$ Poisson (μ), X & Y indept, then $X+Y \sim$ Poisson $(\lambda+\mu)$.

4.6 Randomly stopped sum

Remember the randomly stopped sum model from Section 3.4. A random number N of events occur, and each event i has associated with it a cost or reward X_i. The question is to find the distribution of the total cost or reward: $T_N = X_1 + X_2 + \ldots + X_N$. T_N is called a *randomly stopped sum* because it has a random number of terms.

Example: Cash machine model. N customers arrive during the day. Customer i withdraws amount X_i. The total amount withdrawn during the day is $T_N = X_1 + \ldots + X_N$.

N is RANDOM : this is the difference from what we just did, where n was fixed.

$$G_N(\text{anything}) = \mathbb{E}((\text{anything})^N)$$

So $\quad G_N(G_x(s)) = \mathbb{E}(G_x(s)^N)$

$$= \mathbb{E}(\underbrace{G_x(s) * G_x(s) * \cdots * G_x(s)}_{N \text{ times}})$$

In Chapter 3, we used the Laws of Total Expectation and Variance to show that $\mathbb{E}(T_N) = \mu\,\mathbb{E}(N)$ and $\text{Var}(T_N) = \sigma^2\,\mathbb{E}(N) + \mu^2\,\text{Var}(N)$, where $\mu = \mathbb{E}(X_i)$ and $\sigma^2 = \text{Var}(X_i)$.

In this chapter we will now use probability generating functions to investigate the whole distribution of T_N.

e.g. X_i = money withdrawn by customer i

Theorem 4.6: Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with common PGF G_X. Let N be a random variable, independent of the X_i's, with PGF G_N, and let $T_N = X_1 + \ldots + X_N = \sum_{i=1}^{N} X_i$. Then the PGF of T_N is:

e.g. N = # customers

$$G_{T_N}(s) = G_N\big(G_X(s)\big)$$

Proof:

$$G_{T_N}(s) = \mathbb{E}(s^{T_N})$$

$$= \mathbb{E}\left(s^{X_1 + X_2 + \cdots + X_N}\right) \quad \text{by defns}$$

$$= \mathbb{E}_N\left\{ \mathbb{E}\left(s^{X_1 + \cdots + X_N} \mid N\right) \right\} \quad \text{Law of total expectation}$$

pretend N is constant as long as we're inside the $\{\ \}$.

Potential exam Q (5 marks) write everything in green for exam answer.

$$= \mathbb{E}_N\left\{ \mathbb{E}\left(s^{X_1 + \cdots + X_N}\right) \right\} \quad \text{because } X_i\text{'s are indept of } N$$

(so knowing N fixes # terms in the sum, and then does nothing else to the individual X_i's)

$$= \mathbb{E}_N\left\{ \underbrace{\mathbb{E}(s^{X_1})}_{G_X(s)}\,\underbrace{\mathbb{E}(s^{X_2})}_{G_X(s)} \cdots \underbrace{\mathbb{E}(s^{X_N})}_{G_X(s)} \right\} \quad \text{because } X_i\text{'s indept of each other}$$

$$= \mathbb{E}_N\left\{ \big[G_X(s)\big]^N \right\} \quad \text{because } \mathbb{E}(s^{X_i}) = G_X(s) \text{ for every } i \text{ by defn.}$$

$$= G_N\big(G_X(s)\big) \quad \text{by definition of } G_N. \qquad \square$$

e.g. Ass 3 Q4 d.

Example: Let $X_1, X_2, \ldots$ and N be as above. *use the PGF to* Find the mean of T_N.

$$\mathbb{E}(T_N) = G'_{T_N}(1) = \frac{d}{ds} G_N(G_X(s))\Big|_{s=1}$$

$$= G'_N(G_X(s)) \, G'_X(s) \Big|_{s=1} \qquad \text{chain rule}$$

$$= G'_N(\underbrace{G_X(1)}_{=1 \text{ by §4.2}}) \, \underbrace{G'_X(1)}_{=\mathbb{E}X \text{ by Thm 4.4}} \qquad \text{Subst } s=1$$

$$= G'_N(1) \, \mathbb{E}(X)$$

$$= \mathbb{E}N * \mathbb{E}X \; : \quad \text{Same answer as we found in Section 3.4.}$$

Example: Heron goes fishing

My aunt was asked by her neighbours to feed the prize goldfish in their garden pond while they were on holiday. Although my aunt dutifully went and fed them every day, she never saw a single fish for the whole three weeks. It turned out that all the fish had been eaten by a heron when she wasn't looking!
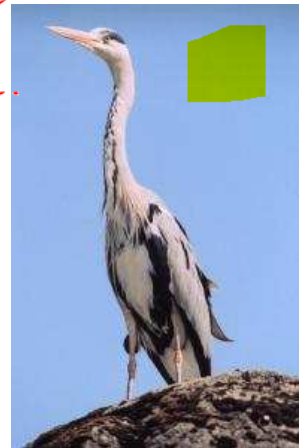
Let N be the number of times the heron visits the pond during the neighbours' absence. Suppose that $N \sim \text{Geometric}(1-\theta)$, so $\mathbb{P}(N=n) = (1-\theta)\theta^n$, for $n = 0, 1, 2, \ldots$. When the heron visits the pond it has probability p of catching a prize goldfish, independently of what happens on any other visit. (This assumes that there are infinitely many goldfish to be caught!) Find the distribution of

$$T = \text{total number of goldfish caught.}$$

Similar to Ass 3 Q1 (a) to (e)

use PGF

Solution:

Let $X_i = \begin{cases} 1 & \text{if heron catches a fish on visit } i \\ 0 & \text{otherwise.} \end{cases}$

Then $T = \#\text{fish caught}$
$$= X_1 + X_2 + \cdots + X_N :$$
this is a randomly stopped sum, stopped by N (random) terms.

So $\boxed{G_T(s) = G_N(G_X(s))}$ $\circledast$

[right margin diagram]

X_i's: $0 \quad 1 \quad 1 \quad 0 \quad 0$
$X_1=0 \; X_2=1$

$N = \#\text{visits}$
$N \sim \text{Geo}(1-\theta)$
$N = 5$ visits here

$T = X_1 + X_2 + \cdots + X_N = 2$ on this diagram.
T is a randomly stopped sum!

So we need $G_X(s)$ and $G_N(r)$.

Now $G_X(s) = \mathbb{E}(s^X)$

$X_i = \begin{cases} 1 & \text{if fish caught (prob } p) \\ 0 & \text{else (prob } 1-p) \end{cases}$

$= s^0 \, \mathbb{P}(X=0) + s^1 \, \mathbb{P}(X=1)$

$\therefore \; G_X(s) = 1-p + ps$ (a)

Also, $G_N(r)$:   (use r instead of s because we actually want

$G_N(G_X(s)) = G_N(1-p+ps)$. So put $r = 1-p+ps$ eventually.

$G_N(r) = \mathbb{E}(r^N) = \sum_{n=0}^{\infty} r^n \, \mathbb{P}(N=n)$

$= \sum_{n=0}^{\infty} r^n \, (1-\theta)\theta^n$

$= (1-\theta) \sum_{n=0}^{\infty} (\theta r)^n$ ← Geometric series

$= (1-\theta) \cdot \dfrac{1}{1-\theta r}$ as long as $|\theta r| < 1$.

$\therefore \; G_N(r) = \dfrac{1-\theta}{1-\theta r}$ for $|r| < \dfrac{1}{\theta}$.   (b)

Substituting (a) and (b) into (✳):

$G_T(s) = G_N(G_X(s)) = \dfrac{1-\theta}{1-\theta G_X(s)}$ putting $r = G_X(s)$ into (b)

$\Rightarrow G_T(s) = \dfrac{1-\theta}{1-\theta(1-p+ps)}$ by (a)

Question: could this be a Geometric r.v.? We would need some value π such that $G_T(s) = \dfrac{1-\pi}{1-\pi s}$ if $T \sim \text{Geo}(1-\pi)$

Collect terms in s on the bottom, & rescale to make $1 - \pi s$: this will reveal what π is. Then see if the top is indeed $1-\pi$? If so, then T really is Geometric.

So: $G_T(s) = \dfrac{1-\theta}{1-\theta+\theta p - \theta ps} = \dfrac{\frac{1-\theta}{1-\theta+\theta p}}{\frac{1-\theta+\theta p}{1-\theta+\theta p} - \frac{\theta ps}{1-\theta+\theta p}}$

$$\Rightarrow G_T(s) = \cfrac{\left(\cfrac{1-\theta}{1-\theta+\theta p}\right)}{1 - \cfrac{\theta p}{1-\theta+\theta p}s} \quad \longleftarrow \text{question is, is this} = 1-\pi \ ?$$

$\longleftarrow$ this must be our π

$$= \cfrac{\cfrac{1-\theta+\theta p - \theta p}{1-\theta+\theta p}}{1 - \cfrac{\theta p}{1-\theta+\theta p}s}$$

$$= \cfrac{1 - \cfrac{\theta p}{1-\theta+\theta p}}{1 - \cfrac{\theta p}{1-\theta+\theta p} * s}$$

$$\Rightarrow G_T(s) = \cfrac{1-\pi}{1-\pi s} \qquad \text{where} \quad \pi = \cfrac{\theta p}{1-\theta+\theta p}.$$

This is the PGF of the Geometric $(1-\pi)$, so by uniqueness of the PGFs, we have discovered $T \sim$ Geometric$(1-\pi) \sim$ Geo$\left(\cfrac{1-\theta}{1-\theta+\theta p}\right)$

$\left.\begin{array}{l} N \sim \text{Geo}(1-\theta) \\ X_i \sim \text{Bernoulli}(p) \end{array}\right\} \Rightarrow T = X_1 + \cdots + X_N \sim \text{Geo}\left(\cfrac{1-\theta}{1-\theta+\theta p}\right).$

Why did we need to use the PGF?   Read (skim)

We could have solved the heron problem without using the PGF, but it is much more difficult. PGFs are very useful for dealing with sums of random variables, which are difficult to tackle using the standard probability function.

Here are the first few steps of solving the heron problem without the PGF. Recall the problem:

- Let $N \sim$ Geometric$(1 - \theta)$, so $\mathbb{P}(N = n) = (1 - \theta)\theta^n$;

- Let $X_1, X_2, \ldots$ be independent of each other and of N, with $X_i \sim$ Binomial$(1, p)$ (remember $X_i = 1$ with probability p, and 0 otherwise);

- Let $T = X_1 + \ldots + X_N$ be the randomly stopped sum;

- Find the distribution of T.

Without using the PGF, we would tackle this by looking for an expression for $\mathbb{P}(T = t)$ for any t. Once we have obtained that expression, we might be able to see that T has a distribution we recognise (e.g. Geometric), or otherwise we would just state that T is defined by the probability function we have obtained.

To find $\mathbb{P}(T = t)$, we have to *partition over different values of N:*

$$\mathbb{P}(T = t) = \sum_{n=0}^{\infty} \mathbb{P}(T = t \,|\, N = n)\mathbb{P}(N = n). \qquad (\star)$$

Here, we are *lucky* that we can write down the distribution of $T \,|\, N = n$:

- if $N = n$ is fixed, then $T = X_1 + \ldots + X_n$ is a sum of n independent Binomial$(1, p)$ random variables, so $(T \,|\, N = n) \sim$ **Binomial**(n, p).

For most distributions of X, *it would be difficult or impossible to write down the distribution of $X_1 + \ldots + X_n$:*

Convolution !

we would have to use an expression like

$$\mathbb{P}(X_1 + \ldots + X_N = t \,|\, N = n) = \sum_{x_1=0}^{t}\sum_{x_2=0}^{t-x_1}\ldots\sum_{x_{n-1}=0}^{t-(x_1+\ldots+x_{n-2})} \Big\{ \mathbb{P}(X_1 = x_1)\times$$

$$\mathbb{P}(X_2 = x_2) \times \ldots \times \mathbb{P}(X_{n-1} = x_{n-1}) \times \mathbb{P}[X_n = t - (x_1 + \ldots + x_{n-1})] \Big\}.$$

Back to the heron problem: we are lucky in this case that we know the distribution of $(T \,|\, N = n)$ is Binomial$(N = n,\, p)$, so

$$\mathbb{P}(T = t \,|\, N = n) = \binom{n}{t}p^t(1-p)^{n-t} \quad \text{for } t = 0, 1, \ldots, n.$$

Continuing from $(\star)$:

$$\mathbb{P}(T = t) \;=\; \sum_{n=0}^{\infty} \mathbb{P}(T = t \,|\, N = n)\mathbb{P}(N = n)$$

$$= \sum_{n=t}^{\infty} \binom{n}{t} p^t (1-p)^{n-t} (1-\theta)\theta^n$$

$$= (1-\theta) \left(\frac{p}{1-p}\right)^t \sum_{n=t}^{\infty} \binom{n}{t} \left[\theta(1-p)\right]^n \qquad (\star\star)$$

$$= \ldots ?$$

As it happens, we can evaluate the sum in $(\star\star)$ using the fact that Negative Binomial probabilities sum to 1. You can try this if you like, but it is quite tricky. [Hint: use the Negative Binomial $(t+1, 1-\theta(1-p))$ distribution.]

Overall, we obtain the same answer that $T \sim$ Geometric $\left(\dfrac{1-\theta}{1-\theta+\theta p}\right)$, but hopefully you can see why the PGF is so useful.

Without the PGF, we have two major difficulties:

1. *Writing down* $\mathbb{P}(T = t \mid N = n)$;

2. *Evaluating the sum over* n *in* $(\star\star)$.

For a general problem, both of these steps might be too difficult to do without a computer. The PGF has none of these difficulties, and even if $G_T(s)$ does not simplify readily, it still tells us everything there is to know about the distribution of T.

4.7 Summary: Properties of the PGF

Definition:	$G_X(s) = \mathbb{E}(s^X)$
Used for:	Discrete r.v.s with values $0, 1, 2, \ldots$
Moments:	$\mathbb{E}(X) = G'_X(1)$ $\qquad$ $\mathbb{E}\left\{X(X-1)\ldots(X-k+1)\right\} = G_X^{(k)}(1)$
Probabilities:	$\mathbb{P}(X = n) = \dfrac{1}{n!} G_X^{(n)}(0)$
Sums:	$G_{X+Y}(s) = G_X(s)G_Y(s)$ for independent X, Y

4.8 Convergence of PGFs Non-Examinable

We have been using PGFs throughout this chapter without paying much attention to their mathematical properties. For example, are we sure that the power series $G_X(s) = \sum_{x=0}^{\infty} s^x \mathbb{P}(X = x)$ converges? Can we differentiate and integrate the infinite power series term by term as we did in Section 4.4? When we said in Section 4.4 that $\mathbb{E}(X) = G_X'(1)$, can we be sure that $G_X(1)$ and its derivative $G_X'(1)$ even exist?

This technical section introduces the **radius of convergence** of the PGF. Although it isn't obvious, it is always safe to assume convergence of $G_X(s)$ at least for $|s| < 1$. Also, there are results that assure us that $\mathbb{E}(X) = G_X'(1)$ will work for all non-defective random variables X.

Definition: The **radius of convergence** of a probability generating function is *a number $R > 0$, such that the sum $G_X(s) = \sum_{x=0}^{\infty} s^x \mathbb{P}(X = x)$ converges if $|s| < R$ and diverges ($\to \infty$) if $|s| > R$.*

(No general statement is made about what happens when $|s| = R$.)

Fact: For any PGF, the radius of convergence exists.

It is always ≥ 1: every PGF converges for at least $s \in (-1, 1)$.

The radius of convergence could be anything from $R = 1$ to $R = \infty$.

Note: This gives us the surprising result that the set of s for which the PGF $G_X(s)$ converges is symmetric about 0: the PGF converges for all $s \in (-R, R)$, and for no $s < -R$ or $s > R$.

This is surprising because the PGF itself is not usually symmetric about 0: i.e. $G_X(-s) \neq G_X(s)$ in general.

Example 1: **Geometric distribution**

Let $X \sim \text{Geometric}(p = 0.8)$. What is the radius of convergence of $G_X(s)$?

As in Section 4.2,

$$G_X(s) = \sum_{x=0}^{\infty} s^x(0.8)(0.2)^x \;\; = \;\; 0.8\sum_{x=0}^{\infty}(0.2s)^x$$

$$= \;\; \frac{0.8}{1-0.2s} \quad \textit{for all } s \textit{ such that } |0.2s| < 1.$$

This is valid for all s with $|0.2s| < 1$, so it is valid for all s with $|s| < \frac{1}{0.2} = 5$.
(i.e. $-5 < s < 5$.)
The radius of convergence is $R = 5$.

The figure shows the PGF of the Geometric($p = 0.8$) distribution, with its radius of convergence $R = 5$. Note that although the convergence set $(-5, 5)$ is symmetric about 0, the function $G_X(s) = p/(1 - qs) = 4/(5 - s)$ is not.

Geometric(0.8) probability generating function
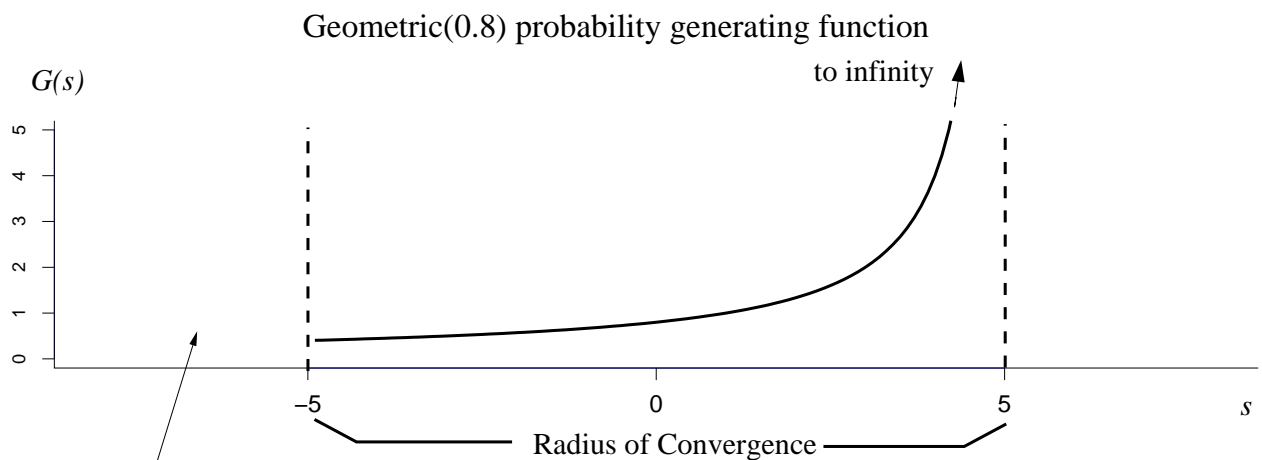
In this region, p/(1−qs) remains finite and well−behaved,
but it is no longer equal to E(s^X).

At the limits of convergence, strange things happen:

- At the positive end, as $s \uparrow 5$, both $G_X(s)$ and $p/(1 - qs)$ approach infinity. So the PGF *is* (left)-continuous at $+R$:

$$\lim_{s\uparrow 5} G_X(s) = G_X(5) = \infty.$$

However, the PGF does *not* converge at $s = +R$.

- At the negative end, as $s \downarrow -5$, the function $p/(1 - qs) = 4/(5 - s)$ is continuous and passes through 0.4 when $s = -5$. However, when $s \le -5$, this function no longer represents $G_X(s) = 0.8 \sum_{x=0}^{\infty}(0.2s)^x$, because $|0.2s| \ge 1$.

  Additionally, when $s = -5$, $G_X(-5) = 0.8 \sum_{x=0}^{\infty}(-1)^x$ does not exist. Unlike the positive end, this means that $G_X(s)$ is *not* (right)-continuous at $-R$:
  $$\lim_{s \downarrow -5} G_X(s) = 0.4 \ne G_X(-5).$$

  Like the positive end, this PGF does *not* converge at $s = -R$.

Example 2: **Binomial distribution**

Let $X \sim \text{Binomial}(n, p)$. What is the radius of convergence of $G_X(s)$?

As in Section 4.2,

$$\begin{aligned}
G_X(s) &= \sum_{x=0}^{n} s^x \binom{n}{x} p^x q^{n-x} \\
&= \sum_{x=0}^{n} \binom{n}{x} (ps)^x q^{n-x} \\
&= (ps + q)^n \quad \text{by the Binomial Theorem: true for all } s.
\end{aligned}$$

This is true for all $-\infty < s < \infty$, so the radius of convergence is $R = \infty$.

Abel's Theorem for continuity of power series at $s = 1$

Recall from above that if $X \sim \text{Geometric}(0.8)$, then $G_X(s)$ is not continuous at the negative end of its convergence $(-R)$:
$$\lim_{s \downarrow -5} G_X(s) \ne G_X(-5).$$

Abel's theorem states that this sort of effect can never happen at $s = 1$ (or at $+R$). In particular, $G_X(s)$ is always left-continuous at $s = 1$:
$$\lim_{s \uparrow 1} G_X(s) = G_X(1) \quad \text{always, even if } G_X(1) = \infty.$$

Theorem 4.8: <u>Abel's Theorem</u>.

$$\text{Let} \quad G(s) = \sum_{i=0}^{\infty} p_i s^i \quad \text{for any } p_0, p_1, p_2, \ldots \text{ with } p_i \geq 0 \text{ for all } i.$$

Then $G(s)$ is left-continuous at $s = 1$:

$$\lim_{s \uparrow 1} G(s) = \sum_{i=0}^{\infty} p_i \; = \; G(1) ,$$

whether or not this sum is finite.

Note: Remember that the radius of convergence $R \geq 1$ for any PGF, so Abel's Theorem means that even in the worst-case scenario when $R = 1$, we can still trust that the PGF will be continuous at $s = 1$. (By contrast, we can not be sure that the PGF will be continuous at the the lower limit $-R$).

Abel's Theorem means that for any PGF, we can write $G_X(1)$ as shorthand for $\lim_{s \uparrow 1} G_X(s)$.

It also clarifies our proof that $\mathbb{E}(X) = G'_X(1)$ from Section 4.4. If we assume that term-by-term differentiation is allowed for $G_X(s)$ (see below), then the proof on page 81 gives:

$$G_X(s) \; = \; \sum_{x=0}^{\infty} s^x \, p_x,$$

$$\text{so} \quad G'_X(s) \; = \; \sum_{x=1}^{\infty} x s^{x-1} p_x \qquad \text{(term-by-term differentiation: see below).}$$

Abel's Theorem establishes that $\mathbb{E}(X)$ is equal to $\lim_{s \uparrow 1} G'_X(s)$:

$$\mathbb{E}(X) \; = \; \sum_{x=1}^{\infty} x p_x$$

$$= \; G'_X(1)$$

$$= \; \lim_{s \uparrow 1} G'_X(s),$$

because Abel's Theorem applies to $G'_X(s) = \sum_{x=1}^{\infty} x s^{x-1} p_x$, establishing that $G'_X(s)$ is left-continuous at $s = 1$. Without Abel's Theorem, we could not be sure that the limit of $G'_X(s)$ as $s \uparrow 1$ would give us the correct answer for $\mathbb{E}(X)$.

Absolute and uniform convergence for term-by-term differentiation

We have stated that the PGF converges for all $|s| < R$ for some R. In fact, the probability generating function converges *absolutely* if $|s| < R$. Absolute convergence is stronger than convergence alone: it means that the sum of absolute values, $\sum_{x=0}^{\infty} |s^x \mathbb{P}(X = x)|$, also converges. When two series both converge absolutely, the product series also converges absolutely. This guarantees that $G_X(s) \times G_Y(s)$ is absolutely convergent for any two random variables X and Y. This is useful because $G_X(s) \times G_Y(s) = G_{X+Y}(s)$ if X and Y are independent.

The PGF also converges *uniformly* on any set $\{s : |s| \le R'\}$ where $R' < R$. Intuitively, this means that the speed of convergence does not depend upon the value of s. Thus a value n_0 can be found such that for all values of $n \ge n_0$, the *finite* sum $\sum_{x=0}^{n} s^x \mathbb{P}(X = x)$ is *simultaneously* close to the converged value $G_X(s)$, for all s with $|s| \le R'$. In mathematical notation: $\forall \epsilon > 0$, $\exists n_0 \in \mathbb{Z}$ such that $\forall s$ with $|s| \le R'$, and $\forall n \ge n_0$,

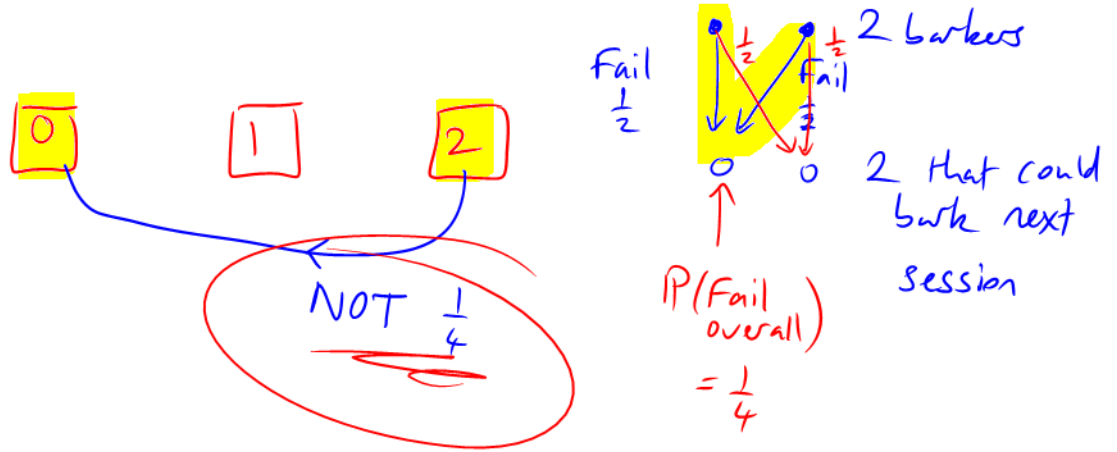$$\left| \sum_{x=0}^{n} s^x \mathbb{P}(X = x) - G_X(s) \right| < \epsilon.$$

Uniform convergence allows us to differentiate or integrate the PGF term by term.

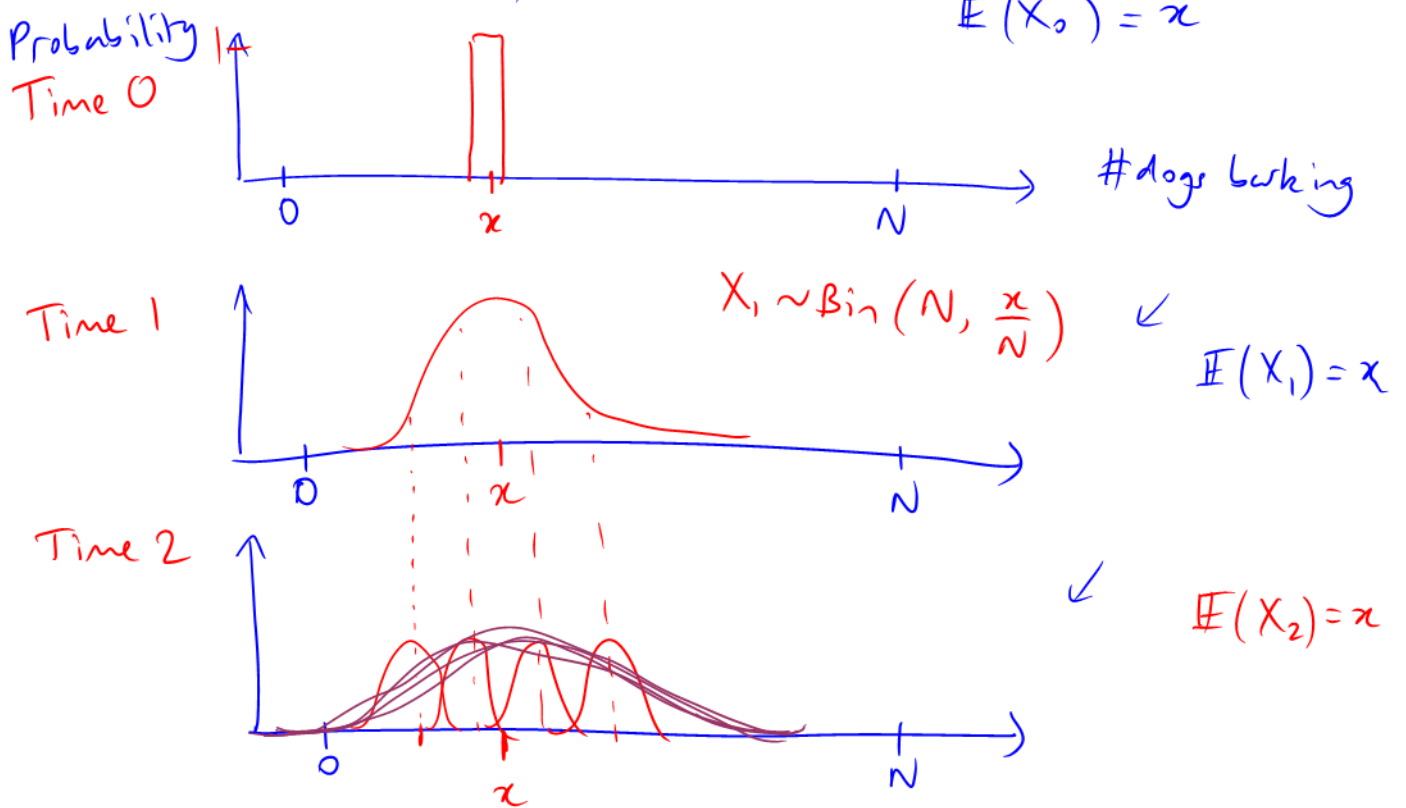**Fact:** Let $G_X(s) = \mathbb{E}(s^X) = \sum_{x=0}^{\infty} s^x \mathbb{P}(X = x)$, and let $s < R$.

1. $G_X'(s) = \dfrac{d}{ds}\left(\sum_{x=0}^{\infty} s^x \mathbb{P}(X = x) \right) = \sum_{x=0}^{\infty} \dfrac{d}{ds}(s^x \mathbb{P}(X = x)) = \sum_{x=0}^{\infty} x s^{x-1} \mathbb{P}(X = x).$
$$\text{(term by term differentiation).}$$

2. $\displaystyle\int_a^b G_X(s)\, ds = \int_a^b \left(\sum_{x=0}^{\infty} s^x \mathbb{P}(X = x) \right) ds = \sum_{x=0}^{\infty} \left(\int_a^b s^x \mathbb{P}(X = x)\, ds \right)$
$$= \sum_{x=0}^{\infty} \frac{s^{x+1}}{x+1} \mathbb{P}(X = x) \quad \text{for} \ -R < a < b < R.$$
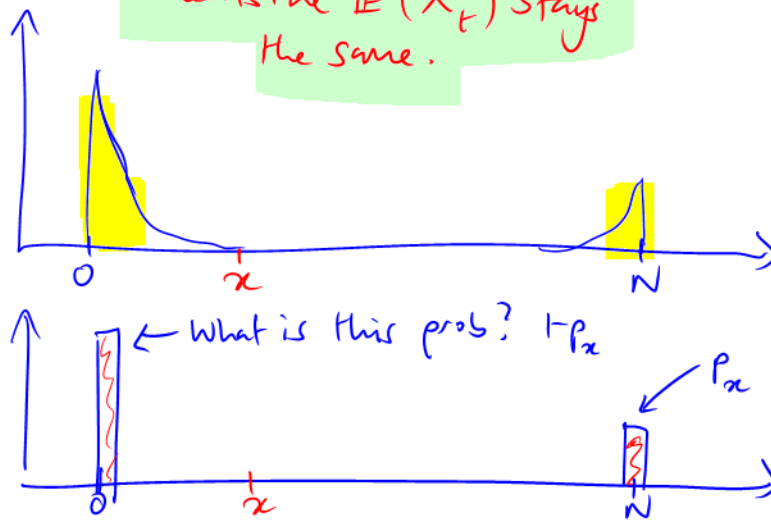$$\text{(term by term integration).}$$

Ass 2 Q 3b

0   1   2

Fail $\frac{1}{2}$ ... $\frac{1}{2}$ Fail   2 barkers

0   0   2 that could bark next session

$\mathbb{P}(\text{Fail overall}) = \frac{1}{4}$

NOT $\frac{1}{4}$

Ass 2 Q 4 (d → h)

$\mathbb{E}(X_0) = x$

Probability
Time 0

\# dogs barking

Time 1

$X_1 \sim Bin\left(N, \frac{x}{N}\right)$

$\mathbb{E}(X_1) = x$

Time 2

$\mathbb{E}(X_2) = x$

Martingale
means the $\mathbb{E}(X_t)$ stays the same.

Time Big

$\mathbb{E}(X_{Big}) = x$

Time ∞

← What is this prob? $1 - P_x$

P_x $\mathbb{E}(X_\infty) = x$

tells us what P_x is.

always on exam.

4.9 Special Process: the Random Walk

We briefly saw the Drunkard's Walk in Chapter 1: a drunk person staggers to left and right as he walks. This process is called the **Random Walk** in stochastic processes. Probability generating functions are particularly useful for processes such as the random walk, because the process is defined as the sum of a single repeating step. The repeating step is a move of one unit, left or right at random. The sum of the first t steps gives the position at time t.

The transition diagram below shows the symmetric random walk (all transitions have probability $p = \frac{1}{2}$).
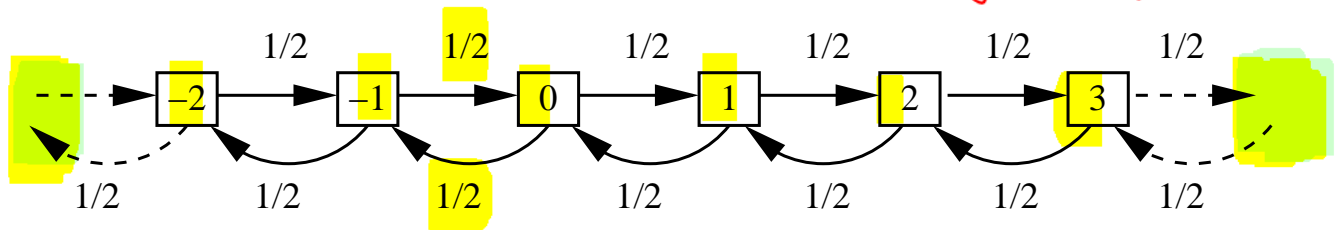
Question:

What is the key difference between the random walk and the gambler's ruin?

The random walk has an INFINITE state space: it never stops. The gambler's ruin stops at both ends.

This fact has two important consequences:

- The random walk is hard to tackle using first-step analysis, because we would have to solve an **infinite** number of simultaneous equations. In this respect it might seem to be more difficult than the gambler's ruin.

- Because the random walk never stops,

  In the gambler's ruin, states are not equal: the states closest to 0 are more likely to end in ruin than the states closest to winning. By contrast, the random walk has no end-points, so (for example) the distribution of the time to reach state 5 starting from state 0 is exactly the same as the distribution of the time to reach state 1005 starting from state 1000. We can exploit this fact to solve some problems for the random walk that would be much more difficult to solve for the gambler's ruin.

$$T = \infty$$

PGFs for finding the distribution of reaching times

For random walks, we are particularly interested in **reaching times:**

- How long will it take us to reach state j, starting from state i?

- Is there a chance that we will **never** reach state j, starting from state i?

In Chapter 3 we saw how to find **expected reaching times:** the expected number of steps taken to reach a particular state. We used the law of total expectation and first-step analysis (§3.5).

However, the *expected* or *average* reaching time doesn't tell the whole story. Think back to the model for gene spread in Section 3.7. If there is just one animal out of 100 with the harmful allele, the expected number of generations to fixation is quite large at 10.5: even though the allele will usually die out after one or two generations. The high average is caused by a small chance that the allele will take hold and grow, requiring a very large number of generations before it either dies out or saturates the population. In most stochastic processes, the average is of limited use by itself, without having some idea about the variance and skew / shape of the distribution.

With our tool of PGFs, we can characterise the *whole distribution* of the time T taken to reach a particular state, by finding its PGF. This will give us the mean, variance, and skew by differentiation. In principle the PGF could even give us the full set of probabilities, $\mathbb{P}(T = t)$ for all possible $t = 0, 1, 2, \ldots$, though in practice it may be computationally infeasible to find more than the first few probabilities by repeated differentiation.

However, there is a new and very useful piece of information that the PGF can tell us quickly and easily:

What is the probability that we NEVER reach state j, starting from state i?

For example, imagine that the random walk represents the share value for an investment. The current share price is i dollars, and we might decide to sell when it reaches j dollars. Knowing how long this might take, and whether there is a chance we will never succeed, is fundamental to managing our investment.

To tackle this problem, we define the random variable T to be the time taken (number of steps) to reach state j, starting from state i. We find the PGF of T, and then use the PGF to discover $\mathbb{P}(T = \infty)$. If $\mathbb{P}(T = \infty) > 0$, there is a positive chance that we will NEVER reach state j, starting from state i.

We will see how to determine the probability of never reaching our goal in Section 4.11. First we will see how to calculate the PGF of a reaching time T in the random walk.
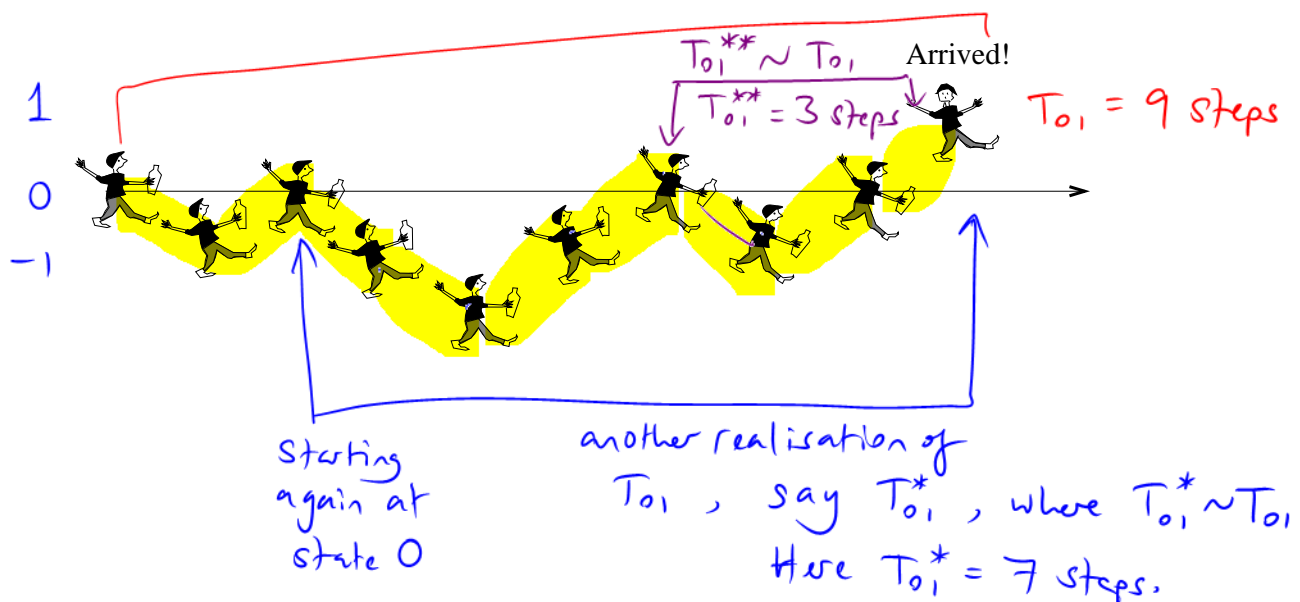
Finding the PGF of a reaching time in the random walk
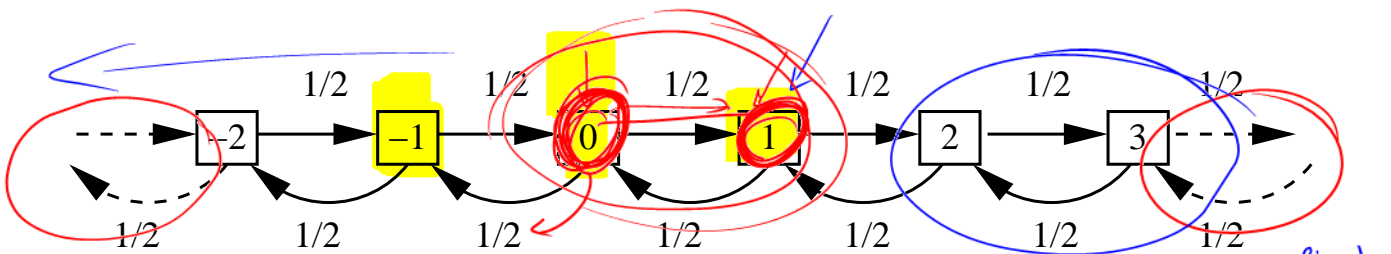
Define T_{ij} to be the number of steps (# arrows crossed) taken to reach state j, starting at state i. (first)

T_{ij} is called the first reaching time from state i to state j.

We will focus on $T_{01} = $ # steps to get from state 0, to state 1 for the first time.

Problem: Let $H(s) = \mathbb{E}\left(s^{T_{01}}\right)$ be the PGF of T_{01}. Find $H(s)$.

Solution: Let Y_n be the step taken at **time n**: either up or down.

$$Y_n = \begin{cases} 1 & \text{w.p. } 0.5 \\ -1 & \text{w.p. } 0.5 \end{cases} \quad \text{(symmetric random walk)}$$

and $Y_1, Y_2, Y_3, \ldots$ are independent.

Recall $T_{ij} = \#$ steps to get from state i to state j for any i, j;

and $H(s) = \mathbb{E}(s^{T_{01}})$ is the PGF required.

Use FSA, partitioning over the first step, Y_1:
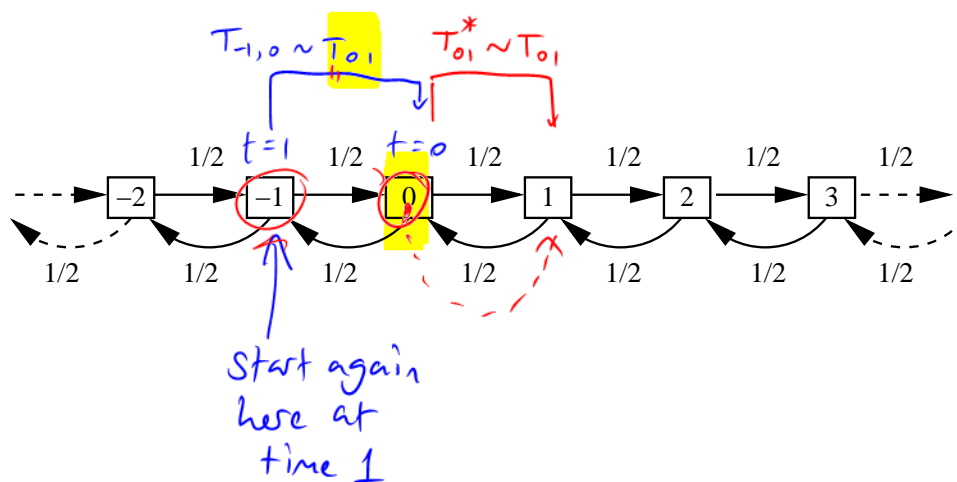
$$H(s) = \mathbb{E}(s^{T_{01}}) \qquad \text{(LoTE)}$$

$$= \mathbb{E}(s^{T_{01}} \mid Y_1 = 1)\, \mathbb{P}(Y_1 = 1) + \mathbb{E}(s^{T_{01}} \mid Y_1 = -1)\, \mathbb{P}(Y_1 = -1)$$

$$\underbrace{}_{\tfrac{1}{2}} \qquad \qquad \underbrace{}_{\tfrac{1}{2}}$$

$$\therefore \quad H(s) = \frac{1}{2} \left\{ \mathbb{E}(s^{T_{01}} \mid Y_1 = 1) + \mathbb{E}(s^{T_{01}} \mid Y_1 = -1) \right\} \quad (\ast)$$

$$\boxed{0} \longrightarrow \boxed{1}$$

If $Y_1 = 1$, then we reached state 1 immediately on the first step, so $T_{01} = 1$ step.

So $\mathbb{E}(s^{T_{01}} \mid Y_1 = 1) = \mathbb{E}(s^1) = s$. (a) First term in $(\ast)$.

$$T_{-1,0} \sim T_{01} \qquad T_{01}^* \sim T_{01}$$

1/2 $t=1$ 1/2 $t=0$ 1/2   1/2   1/2   1/2

$\boxed{-2} \quad \boxed{-1} \quad \boxed{0} \quad \boxed{1} \quad \boxed{2} \quad \boxed{3}$

1/2   1/2   1/2   1/2   1/2   1/2   1/2

Start again here at time 1

For 2nd term of ✪: $\mathbb{E}(s^{T_{01}} \mid Y_1 = -1)$:

If $Y_1 = -1$, then $T_{01} = 1 + T_{-1,1}$

I step from state 0 to state −1

unknown # steps to get back from −1, to original destination of 1, for the first time.

So $\quad T_{01} = 1 + T^*_{-1,0} + T^*_{0,1}$

because to get −1 to 1 for the first time, I must in between reach 0 for the lst time, and then from 0 reach 1 for the first time. So $\quad T^*_{-1,0} \sim T_{01}$ and $\quad T^*_{01} \sim T_{01}$ } both concerned with time taken to move 1 step to the right for the lst time

And, T^*_{-10}, T^*_{01} are independent — because they cover separate time intervals.

So $\mathbb{E}(s^{T_{01}} \mid Y_1 = -1) = \mathbb{E}\left(s^{1 + T^*_{-10} + T^*_{01}} \right)$

$$= \mathbb{E}\left(s^1 \, s^{T^*_{-10}} \, s^{T^*_{01}} \right)$$

$$= s \, \mathbb{E}\left(s^{T^*_{-10}} \right) \mathbb{E}\left(s^{T^*_{01}} \right) \quad \text{by indep of } T^*_{-10} \text{ and } T^*_{01}$$

$$= s \, H(s) \, H(s) \quad \text{because } T^*_{-10} \sim T^*_{01} \sim T_{01}.$$

Subst in ✪: $\quad H(s) = \frac{1}{2}\left\{ s + s\left[H(s)\right]^2 \right\}.$

This is a quadratic in $H(s)$:

$$\frac{1}{2} s \left[H(s)\right]^2 - H(s) + \frac{1}{2}s = 0 \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad H(s) = \frac{1 \pm \sqrt{1 - 4 * \frac{1}{2}s * \frac{1}{2}s}}{s} = \frac{1 \pm \sqrt{1 - s^2}}{s}.$$

Which root? We know $\mathbb{P}(T_{01} = 0) = 0$ because it takes $\geqslant 1$ step to go from 0 to 1. ← property of PGF
So $\mathbb{P}(T_{01} = 0) = H(0) = 0$. Only $(-)$ root allows this.

Overall, $H(s) = \dfrac{1 - \sqrt{1-s^2}}{s}$ ← $f(s)$
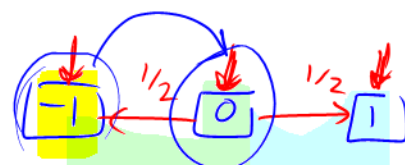
                              ← $g(s)$

Check the $(-)$ root does have the correct limit $\lim\limits_{s\to 0} H(s) = 0$

using L'Hospital's Rule:

$$\lim_{s\to 0} \frac{f(s)}{g(s)} = \lim_{s\to 0} \frac{f'(s)}{g'(s)} = \lim_{s\to 0}\left\{\frac{\frac{1}{2}(1-s^2)^{-1/2}(2s)}{1}\right\} = \frac{0}{1} = 0$$

So we have proved the $(-)$ root DOES give the correct limit,

$$\lim_{s\to 0} H(s) = 0.$$

$\mathbb{P}(T=0) = 0$.

Use This Method /Notation whenever you do Qs on this.

Notation for quick solutions of first-step analysis for finding PGFs

As with first-step analysis for finding hitting probabilities and expected reaching times, setting up a good notation is extremely important. Here is a good notation for finding $H(s) = \mathbb{E}\left(s^{T_{01}}\right)$.

$T_{01} = $ #steps to get from state 0 to state 1 for the first time.

Let $T = T_{01}$. We seek $H(s) = \mathbb{E}(s^T)$.

Now $T = \begin{cases} 1 & \text{with probability } \frac{1}{2} & \leftarrow \text{go right at 1st step} \\ 1 + T' + T'' & \text{with prob } \frac{1}{2} & \leftarrow \text{go left at 1st step} \end{cases}$

1 step to go left to -1

from state -1, need to get to state 0 for 1st time: this has the same distn as T

from state 0 for first time, need to get to state 1 for first time: again, it has the same distn as T

where T' and T'' are independent, and $T' \sim T'' \sim T$.

So $s^T = \begin{cases} s^1 & \text{w.p. } \frac{1}{2} \\ s^{1+T'+T''} & \text{w.p. } \frac{1}{2} \end{cases}$

So $H(s) = \mathbb{E}(s^T) = \frac{1}{2}\mathbb{E}(s^1) + \frac{1}{2}\mathbb{E}\left(s^{1+T'+T''}\right)$

$= \frac{1}{2}s + \frac{1}{2}s\,\mathbb{E}(s^{T'})\mathbb{E}(s^{T''})$ by indep of T', T''

$= \frac{1}{2}s + \frac{1}{2}s\left[H(s)\right]^2$ because $T' \sim T'' \sim T$.

$\Rightarrow H(s) = \frac{1}{2}s + \frac{1}{2}s\left[H(s)\right]^2$

$\Rightarrow s\,H(s)^2 - 2H(s) + s = 0$ (Quadratic in $H(s)$)

Test!

This Wed in lecture hour, Rm **BLT** 204 as usual.

START AT 10am!

Bring own paper, calculator, stapler if poss.

READ INFO SHEET FOR PRACTICE INFO!

Solve the quadratic:

$$H(s) = \frac{1 \,\textcircled{\pm}\, \sqrt{1-s^2}}{s} . \quad \checkmark$$

general property of PGFs

Which root? We know that $H(0) = P(T=0)$

$\Rightarrow \quad H(0) = \underline{0}$ for this r.v. T, because it must take ≥ 1 step to get from state 0 to state 1.

Try the (+) root: $\displaystyle\lim_{s\to 0} H(s) = \lim_{s\to 0} \frac{1+\sqrt{1-s^2}}{s} \rightsquigarrow \frac{2}{0}$ can NOT reach correct limit of 0.

So it CAN'T BE the (+) root, so it MUST be the (−) root.

$$\therefore \quad H(s) = \frac{1 - \sqrt{1-s^2}}{s}$$

(−) root:

$\displaystyle\lim_{s\to 0} \frac{1-\sqrt{1-s^2}}{s} \rightsquigarrow \frac{0}{0}$

COULD have the correct limit of 0. Use L'Hospital's Rule to verify.

4.10 Defective random variables

A random variable is said to be **defective** if it can take the value ∞.

In stochastic processes, a reaching time T_{ij} is defective if there is A CHANCE that we NEVER reach state j, starting from state i.

The probability that we never reach state j, starting from state i, is the same as the probability that the time taken is infinite: $T_{ij} = \infty$:

$$P(T_{ij} = \infty) = P(\text{we NEVER reach state } j, \text{ starting from state } i).$$

In other cases, we will ALWAYS reach state j eventually, starting from state i.

In that case, T_{ij} can NOT take the value ∞.

$\rightarrow P(T_{ij} = \infty) = 0$ if we are CERTAIN to reach state j, starting from state i.

Definition: A random variable T is **defective**, or **improper**, if it can take the value ∞. That is,

$$T \text{ is defective if } P(T = \infty) \;>\; 0 .$$

$$T \text{ is non-defective if } P(T = \infty) = 0.$$

$\sum_{t=0}^{\infty}$ $t = 0, 1, 2, 3, \ldots$
runs through all finite values in $\mathbb{N}$ without stopping.

Thinking of $\sum_{t=0}^{\infty} \mathbb{P}(T = t)$ as $1 - \mathbb{P}(T = \infty)$

Although it seems strange, when we write $\sum_{t=0}^{\infty} \mathbb{P}(T = t)$, we are NOT including the value $t = \infty$. Instead, $\sum_{t=0}^{\infty}$ just means "keep going and never stop".

The sum $\sum_{t=0}^{\infty}$ continues without ever stopping: at no point can we say we have 'finished' all the finite values of t so we will now add on $t = \infty$. We simply never get to $t = \infty$ when we take $\sum_{t=0}^{\infty}$.

For a defective random variable T, this means that

$$\sum_{t=0}^{\infty} \mathbb{P}(T = t) < 1$$

because we are missing the positive value of $\mathbb{P}(T = \infty)$.

All probabilities of T must still sum to 1, so we have

$$1 = \sum_{t=0}^{\infty} \mathbb{P}(T = t) + \mathbb{P}(T = \infty)$$

sum of all probs of finite values $t = 0, 1, 2, \ldots$

extra blob of probability we have to add on to the end.

in other words

$$\sum_{t=0}^{\infty} \mathbb{P}(T = t) = 1 - \mathbb{P}(T = \infty).$$

PGFs for defective random variables

When T is defective, the PGF of T is *defined as* the power series

$H(s) = \mathbb{E}(s^T)$
$= \sum_{t=0}^{\infty} s^t \mathbb{P}(T = t)$

$$H(s) = \sum_{t=0}^{\infty} \mathbb{P}(T = t) s^t \quad \text{for } |s| < 1.$$

The term for $\mathbb{P}(T = \infty) s^\infty$ is missed out. The PGF is defined as the generating function of the probabilities for finite values only.

But note that $s^\infty = 0$ for $|s| < 1$ anyway, so as long as we keep $|s| < 1$, it makes no difference whether we include $s^\infty \mathbb{P}(T = \infty)$ or not.

$$H(s) = \sum_{t=0}^{\infty} s^t \, \mathbb{P}(T = t)$$

$$\mathbb{E}(s^T) = \sum_{t=0}^{\infty} s^t \, \mathbb{P}(T=t) + s^{\infty} \, \mathbb{P}(T=\infty)$$

both the same thing when $|s| < 1$ because $s^{\infty} = 0$ when $|s| < 1$.

When $s=1$, the term $s^{\infty} \mathbb{P}(T=\infty)$ suddenly springs to life and $= \mathbb{P}(T=\infty)$

Because $H(s)$ is a power series satisfying the conditions of Abel's Theorem, we know that:

$$\sum_{t=0}^{\infty} s^t \, \mathbb{P}(T=t)$$

- $H(s)$ is left-continuous at $s = 1$, i.e. $\lim_{s \uparrow 1} H(s) = H(1)$.

This is different from the behaviour of $\mathbb{E}(s^T)$, if T is defective:

- $\mathbb{E}(s^T) = H(s)$ for $|s| < 1$ because the missing term is zero: i.e. because $s^{\infty} = 0$ when $|s| < 1$.

- $\mathbb{E}(s^T)$ is NOT left-continuous at $s = 1$. There is a sudden leap (discontinuity) at $s = 1$ because $s^{\infty} = 0$ as $s \uparrow 1$, but $s^{\infty} = 1$ when $s = 1$.

Thus $H(s)$ does NOT represent $\mathbb{E}(s^T)$ at $s = 1$. It is as if $H(s)$ is a 'train' that $\mathbb{E}(s^T)$ rides on between $-1 < s < 1$. At $s = 1$, the train keeps going (i.e. $H(s)$ is continuous) but $\mathbb{E}(s^T)$ jumps off the train.

We test whether T is defective by testing whether or not $\mathbb{E}(s^T)$ 'jumps off the train' — that is, we test whether or not $H(s)$ is equal to $\mathbb{E}(s^T)$ when $s = 1$.

We **know** what $\mathbb{E}(s^T)$ is when $s = 1$:

- $\mathbb{E}(s^T)$ is always 1 when $s = 1$, whether T is defective or not:

$$\mathbb{E}(1^T) = 1 \qquad \text{for ANY random variable } T.$$

But the function $H(s) = \sum_{t=0}^{\infty} s^t \mathbb{P}(T = t)$ may or may not be 1 when $s = 1$:

- If T is defective, $H(s)$ is missing a term and $H(1) < 1$.

- If T is not defective, $H(s)$ is not missing anything so $H(1) = 1$.

Test for defectiveness:

Let $H(s) = \sum_{t=0}^{\infty} s^t \mathbb{P}(T = t)$ be the power series representing the PGF of T for $|s| < 1$. Then T is defective if and only if $H(1) < 1$.

If $H(1) < 1$, then $\mathbb{P}(T=\infty)$ is the missing piece:

$$\mathbb{P}(T=\infty) = 1 - H(1).$$

$\frac{1}{2}$ $\frac{1}{2}$

$\boxed{-1} \xleftarrow{} \boxed{0} \xrightarrow{} \boxed{1}$

T not defective

Q: will we DEFINITELY Reach state 1, starting from state 0? Or could we drift left forever? (T defective)

Using defectiveness to find the probability we never get there

The simple test for defectiveness tells us whether there is a positive probability that we NEVER reach our goal. Here are the steps.

1. We want to know the probability that we will NEVER reach state j, starting from state i.

2. Define T to be the random variable giving the *number of steps taken* to get from state i to state j.

3. The event that we never reach state j, starting from state i, is the same as the event that $T = \infty$. (If we wait an infinite length of time, we never get there.) So

$$\mathbb{P}(\text{never reach state } j \mid \text{start at state } i) = \mathbb{P}(T = \infty).$$

e.g. $H(s) = \frac{1 - \sqrt{1-s^2}}{s}$ for $|s| < 1$.

4. Find $H(s) = \sum_{t=0}^{\infty} s^t \mathbb{P}(T = t)$, using a calculation like the one we did in Section 4.9. $H(s)$ is the PGF of T for $|s| < 1$. We only need to find it for $|s| < 1$. The calculation in Section 4.9 only works for $|s| \leq 1$ because the expectations are infinite or undefined when $|s| > 1$.

5. The random variable T is defective if and only if $H(1) < 1$.

6. If $H(1) < 1$, then the probability that T takes the value ∞ is the missing piece: $\mathbb{P}(T = \infty) = 1 - H(1)$.

Use this in calculations (Ass 3)

Overall:

$$\boxed{\mathbb{P}(\text{never reach state } j \mid \text{start at state } i) = \mathbb{P}(T = \infty) = 1 - H(1)}$$

Expectation and variance of a defective random variable

If T is defective, there is a positive chance that $T = \infty$. This means that

$$\mathbb{E}(T) = \infty \quad \text{and} \quad \text{Var}(T) = \infty, \quad \text{and} \quad \mathbb{E}(T^a) = \infty$$

for any power a.

$$\mathbb{E}(T) = \sum_{t=0}^{\infty} t\, \mathbb{P}(T = t) + \infty\, \mathbb{P}(T = \infty)$$

So if $\mathbb{P}(T = \infty) > 0$ then $\mathbb{E}T = \infty$.

$\mathbb{E}(T)$ and $\text{Var}(T)$ can not be found using the PGF when T is defective: you will get the wrong answer.

When you are asked to find $\mathbb{E}(T)$ in a context where T might be defective:

$\longrightarrow$ • First check whether T is defective: is $H(1) < 1$ (defective) or $H(1) = 1$
                                                             (not defective)

—— • If T is defective, then $\mathbb{E}(T) = \infty$.

✓ • If T is not defective ($H(1) = 1$), then $\mathbb{E}(T) = H'(1)$ as usual.

4.11 Random Walk: the probability we never reach our goal

In the random walk in Section 4.9, we defined the first reaching time T_{01} as the number of steps taken to get from state 0 to state 1.

In Section 4.9 we found the PGF of T_{01} to be:

$$\text{PGF of } T_{01} \quad = \quad H(s) \quad = \quad \frac{1 - \sqrt{1 - s^2}}{s} \qquad \text{for } |s| < 1.$$

Questions:

a) What is the probability that we *never* reach state 1, starting from state 0?

b) What is expected number of steps to reach state 1, starting from state 0?

Solutions: a) $P(\text{never reach state 1} \mid \text{start from state 0}) = 1 - H(1)$

$$= 1 - \left\{ \frac{1 - \sqrt{1 - 1^2}}{1} \right\}$$

∴ $P\left(\text{never reach state 1} \mid \substack{\text{start} \\ \text{at} \\ 0}\right) = 0$

So T is NOT defective, and we DEFINITELY will reach state 1 eventually, even if it takes a very long time.

$\longrightarrow$ Consequence: we will definitely reach everywhere in that case! [Mindblowing!]

b) Because T_{01} is not defective, we can find $\mathbb{E}(T_{01})$ by differentiating the PGF at $s=1$:

$$\mathbb{E}(T_{01}) = H'(1)$$

Now $\quad H(s) = \dfrac{1 - \sqrt{1-s^2}}{s} = s^{-1} - \left(s^{-2} - 1\right)^{1/2}$

$\sqrt{\dfrac{1-s^2}{s^2}}$ $\quad$ So $\quad H'(s) = -s^{-2} - \dfrac{1}{2}\left(s^{-2} - 1\right)^{-1/2}\left(-2s^{-3}\right)$

$$\underset{s=1}{\underbrace{}}$$

$$\Rightarrow (1-1)^{-1/2}$$

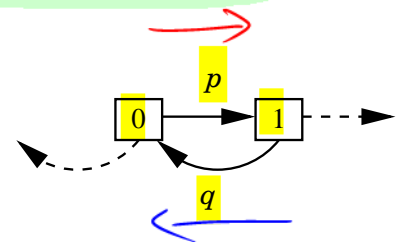$$\Rightarrow \frac{1}{\sqrt{0}}$$

$$\Rightarrow \quad H'(1) = \lim_{s \uparrow 1} H'(s) = \infty .$$

So $\quad \mathbb{E}(T_{01}) = \infty \quad$!!

This result is striking. Even though we will **definitely** reach state 1, the expected time to do so is infinite! In general, we can prove the following results for random walks, starting from state 0:

Property	Reach state 1?	$\mathbb{P}(T_{01} = \infty)$	$\mathbb{E}(T_{01})$
$p > q$	$\rightarrow$ Guaranteed ✓	0	finite ✓
$p = q = \frac{1}{2}$	Guaranteed	0	∞
$p < q$	Not guaranteed ✓	> 0 T_{01} defective	∞ ✓

Note: (**Non-examinable**) If T is defective in the random walk, $\mathbb{E}(s^T)$ is not continuous at $s = 1$. In Section 4.9 we had to solve a quadratic equation to find $H(s) = \mathbb{E}(s^T)$. The negative root solution for $H(s)$ generally represents $\mathbb{E}(s^T)$ for $s < 1$. At $s = 1$, the solution for $\mathbb{E}(s^T)$ suddenly flips from the $-$ root to the $+$ root of the quadratic. This explains how $\mathbb{E}(s^T)$ can be discontinuous as $s \uparrow 1$, even though the negative root for $H(s)$ is continuous as $s \uparrow 1$ and all the working of Section 4.9 still applies for $s = 1$. The reason is that we suddenly switch from the $-$ root to the $+$ root at $s = 1$.

When $|s| > 1$, the conditional expectations are not finite so the working of Section 4.9 no longer applies.