

Chapter 5: Mathematical Induction

So far in this course, we have seen some techniques for dealing with stochastic processes: first-step analysis for hitting probabilities (Chapter 2), and first-step analysis for expected reaching times (Chapter 3). We now look at another tool that is often useful for exploring properties of stochastic processes: *proof by mathematical induction*.

5.1 Proving things in mathematics

There are many different ways of constructing a formal proof in mathematics. Some examples are:

- **Proof by counterexample:** a proposition is proved to be *not generally true* because a *particular example* is found for which it is not true.
- **Proof by contradiction:** this can be used either to prove a proposition is true or to prove that it is false. To prove that the proposition is *true* (say), we start by *assuming that it is false*. We then explore the consequences of this assumption until we reach a contradiction, e.g. $0 = 1$. Therefore something must have gone wrong, and the only thing we weren't sure about was our initial assumption that the proposition is false — so our initial assumption must be wrong and the proposition is proved true.

A famous proof of this sort is the proof that there are infinitely many prime numbers. We start by assuming that there are *finitely* many primes, so they can be listed as p_1, p_2, \dots, p_n , where p_n is the largest prime number. But then the number $p_1 \times p_2 \times \dots \times p_n + 1$ must also be prime, because it is not divisible by any of the smaller primes. Furthermore this number is definitely bigger than p_n . So we have contradicted the idea that there was a 'biggest' prime called p_n , and therefore there are infinitely many primes.

- **Proof by mathematical induction:** in mathematical induction, we start with a formula that we *suspect* is true. For example, I might *suspect* from

observation that $\sum_{k=1}^n k = n(n+1)/2$. I might have tested this formula for many different values of n , but of course I can never test it for *all* values of n . Therefore I need to prove that the formula is *always* true.

The idea of mathematical induction is to say: *suppose* the formula is true for all n up to the value $n = 10$ (say). Can I prove that, *if* it is true for $n = 10$, *then* it will also be true for $n = 11$? And *if* it is true for $n = 11$, then it will also be true for $n = 12$? And so on.

In practice, we usually start lower than $n = 10$. We usually take the very easiest case, $n = 1$, and prove that the formula is true for $n = 1$: $\text{LHS} = \sum_{k=1}^1 k = 1 = 1 \times 2/2 = \text{RHS}$. Then we prove that, *if* the formula is ever true for $n = x$, *then* it will always be true for $n = x + 1$. Because it is true for $n = 1$, it must be true for $n = 2$; and because it is true for $n = 2$, it must be true for $n = 3$; and so on, for all possible n . Thus the formula is proved.

Mathematical induction is therefore a bit like a *first-step analysis for proving things: prove that wherever we are now, the next step will always be OK. Then if we were OK at the very beginning, we will be OK for ever.*

The method of mathematical induction for proving results is very important in the study of Stochastic Processes. This is because a stochastic process builds up one step at a time, and mathematical induction works on the same principle.

Example: We have already seen examples of inductive-type reasoning in this course. For example, in Chapter 2 for the Gambler's Ruin problem, using the method of repeated substitution to solve for $p_x = \mathbb{P}(\text{Ruin} \mid \text{start with } \$x)$, we discovered that:

- $p_2 = 2p_1 - 1$
- $p_3 = 3p_1 - 2$
- $p_4 = 4p_1 - 3$

We deduced that $p_x = xp_1 - (x - 1)$ *in general*.

To prove this properly, we should have used the method of mathematical induction.

5.2 Mathematical Induction by example

This example explains the style and steps needed for a proof by induction.

Question: Prove by induction that $\sum_{k=1}^n k = \frac{n(n+1)}{2}$ for any integer n . (\star)

Approach: follow the steps below.

- (i) First verify that the formula is true for a *base case*: usually the smallest appropriate value of n (e.g. $n = 0$ or $n = 1$). Here, the smallest possible value of n is $n = 1$, because we can't have $\sum_{k=1}^0$.

First verify (\star) is true when $n = 1$.

$$LHS = \sum_{k=1}^1 k = 1.$$

$$RHS = \frac{1 \times 2}{2} = 1 = LHS.$$

So (\star) is proved for $n = 1$.

- (ii) Next suppose that formula (\star) is true for *all values of n up to and including some value x* . (We have already established that this is the case for $x = 1$).

Using the hypothesis that (\star) is true for all values of n up to and including x , prove that it is therefore true for the value $n = x + 1$.

Now suppose that (\star) is true for $n = 1, 2, \dots, x$ for some x .

Thus we can assume that $\sum_{k=1}^x k = \frac{x(x+1)}{2}$. (a)

((a) for 'allowed' info)

We need to show that if (\star) holds for $n = x$, then it must also hold for $n = x + 1$.

Require to prove that

$$\sum_{k=1}^{x+1} k = \frac{(x+1)(x+2)}{2} \quad (**)$$

(Obtained by putting $n = x + 1$ in $(*)$).

$$\begin{aligned} \text{LHS of } (**) &= \sum_{k=1}^{x+1} k = \sum_{k=1}^x k + (x+1) && \text{by expanding the sum} \\ &= \frac{x(x+1)}{2} + (x+1) && \text{using allowed info (a)} \\ &= (x+1) \left(\frac{x}{2} + 1 \right) && \text{rearranging} \\ &= \frac{(x+1)(x+2)}{2} \\ &= \text{RHS of } (**). \end{aligned}$$

This shows that:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2} \quad \text{when } n = x + 1.$$

So, assuming $(*)$ is true for $n = x$, it is also true for $n = x + 1$.

- (iii) Refer back to the base case: if it is true for $n = 1$, then it is true for $n = 1+1 = 2$ by (ii). If it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ by (ii). We could go on forever. This proves that the formula $(*)$ is true for all n .

We proved $(*)$ true for $n = 1$, thus $(*)$ is true for all integers $n \geq 1$. □

General procedure for proof by induction

The procedure above is quite standard. The inductive proof can be summarized like this:

Question: prove that $f(n) = g(n)$ for all integers $n \geq 1$. (\star)

Base case: $n = 1$. Prove that $f(1) = g(1)$ using

$$\begin{aligned} LHS &= f(1) \\ &= \vdots \\ &= g(1) = RHS. \end{aligned}$$

General case: suppose (\star) is true for $n = x$:

$$\text{so } f(x) = g(x). \quad (a) \quad (\text{allowed info})$$

Prove that (\star) is therefore true for $n = x + 1$:

$$\text{RTP } f(x + 1) = g(x + 1). \quad (\star\star)$$

$$\begin{aligned} LHS(\star\star) &= f(x + 1) \\ &= \left\{ \begin{array}{l} \text{some expression breaking down } f(x + 1) \\ \text{into } f(x) \text{ and an extra term in } x + 1 \end{array} \right\} \\ &= \left\{ \text{substitute } f(x) = g(x) \text{ in the line above} \right\} \quad \text{by allowed (a)} \\ &= \{ \text{do some working} \} \\ &= g(x + 1) \\ &= RHS(\star\star). \end{aligned}$$

Conclude: (\star) is proved for $n = 1$, so it is proved for $n = 2, n = 3, n = 4, \dots$

(\star) is therefore proved for all integers $n \geq 1$. \square

5.3 Some harder examples of mathematical induction

Induction problems in stochastic processes are often trickier than usual. Here are some possibilities:

- Backwards induction: start with base case $n = N$ and go backwards, instead of starting at base case $n = 1$ and going forwards.
- Two-step induction, where the proof for $n = x + 1$ relies not only on the formula being true for $n = x$, but also on it being true for $n = x - 1$.

The first example below is hard probably because it is too easy. The second example is an example of a two-step induction.

Example 1: Suppose that $p_0 = 1$ and $p_x = \alpha p_{x+1}$ for all $x = 1, 2, \dots$. Prove by mathematical induction that $p_n = 1/\alpha^n$ for $n = 0, 1, 2, \dots$

Wish to prove

$$p_n = \frac{1}{\alpha^n} \quad \text{for } n = 0, 1, 2, \dots \quad (\star)$$

Information given:

$$p_{x+1} = \frac{1}{\alpha} p_x \quad (G_1)$$

$$p_0 = 1 \quad (G_2)$$

Base case: $n = 0$.

LHS = $p_0 = 1$ by information given (G_2).

$$RHS = \frac{1}{\alpha^0} = \frac{1}{1} = 1 = LHS.$$

Therefore (\star) is true for the base case $n = 0$.

General case: suppose that (\star) is true for $n = x$, so we can assume

$$p_x = \frac{1}{\alpha^x}. \quad (a)$$

Wish to prove that (\star) is also true for $n = x + 1$: i.e.

$$RTP \quad p_{x+1} = \frac{1}{\alpha^{x+1}}. \quad (\star\star)$$

$$\begin{aligned}
 \text{LHS of } (**) = p_{x+1} &= \frac{1}{\alpha} \times p_x && \text{by given } (G_1) \\
 &= \frac{1}{\alpha} \times \frac{1}{\alpha^x} && \text{by allowed } (a) \\
 &= \frac{1}{\alpha^{x+1}} \\
 &= \text{RHS of } (**).
 \end{aligned}$$

So if formula $(*)$ is true for $n = x$, it is true for $n = x + 1$. We have shown it is true for $n = 0$, so it is true for all $n = 0, 1, 2, \dots$ \square

Example 2: Gambler's Ruin. In the Gambler's Ruin problem in Section 2.7, we have the following situation:

- $p_x = \mathbb{P}(\text{Ruin} \mid \text{start with } \$x)$;
- We know from first-step analysis that $p_{x+1} = 2p_x - p_{x-1}$ (G_1)
- We know from common sense that $p_0 = 1$ (G_2)
- By direct substitution into (G_1) , we obtain:

$$\begin{aligned}
 p_2 &= 2p_1 - 1 \\
 p_3 &= 3p_1 - 2
 \end{aligned}$$

- We develop a suspicion that for all $x = 1, 2, 3, \dots$,

$$p_x = xp_1 - (x - 1) \quad (*)$$

- We wish to prove $(*)$ by mathematical induction.

For this example, *our given information, in (G_1) , expresses p_{x+1} in terms of both p_x and p_{x-1} , so we need two base cases. Use $x = 1$ and $x = 2$.*

Wish to prove $p_x = xp_1 - (x - 1)$ (\star).

Base case $x = 1$:

$$LHS = p_1.$$

$$RHS = 1 \times p_1 - 0 = p_1 = LHS.$$

\therefore formula (\star) is true for base case $x = 1$.

Base case $x = 2$:

$$LHS = p_2 = 2p_1 - 1 \quad \text{by information given } (G_1)$$

$$RHS = 2 \times p_1 - 1 = LHS.$$

\therefore formula (\star) is true for base case $x = 2$.

General case: suppose that (\star) is true for all x up to $x = k$.

So we are allowed:

$$\begin{array}{lll} (x = k) & p_k = kp_1 - (k - 1) & (a_1) \\ (x = k - 1) & p_{k-1} = (k - 1)p_1 - (k - 2) & (a_2) \end{array}$$

Wish to prove that (\star) is also true for $x = k + 1$, i.e.

$$RTP \quad p_{k+1} = (k + 1)p_1 - k. \quad (\star\star)$$

$$\begin{aligned} LHS \text{ of } (\star\star) &= p_{k+1} \\ &= 2p_k - p_{k-1} \quad \text{by given information } (G_1) \\ &= 2 \left\{ kp_1 - (k - 1) \right\} - \left\{ (k - 1)p_1 - (k - 2) \right\} \\ &\quad \text{by allowed } (a_1) \text{ and } (a_2) \\ &= p_1 \left\{ 2k - (k - 1) \right\} - \left\{ 2(k - 1) - (k - 2) \right\} \\ &= (k + 1)p_1 - k \\ &= RHS \quad \text{of } (\star\star) \end{aligned}$$

So if formula (\star) is true for $x = k - 1$ and $x = k$, it is true for $x = k + 1$. We have shown it is true for $x = 1$ and $x = 2$, so it is true for all $x = 1, 2, 3, \dots$ \square