Chapter 5: Mathematical Induction

So far in this course, we have seen some techniques for dealing with stochastic processes: first-step analysis for hitting probabilities (Chapter 2), and first-step analysis for expected reaching times (Chapter 3). We now look at another tool that is often useful for exploring properties of stochastic processes: proof by mathematical induction.

5.1 Proving things in mathematics

There are many different ways of constructing a formal proof in mathematics. Some examples are:

- **Proof by counterexample:** a proposition is proved to be *not generally true* because a *particular example* is found for which it is not true.
- **Proof by contradiction:** this can be used either to prove a proposition is true or to prove that it is false. To prove that the proposition is *true* (say), we start by *assuming that it is false.* We then explore the consequences of this assumption until we reach a contradiction, e.g. $0 = 1$. Therefore something must have gone wrong, and the only thing we weren't sure about was our initial assumption that the proposition is false — so our initial assumption must be wrong and the proposition is proved true.

  A famous proof of this sort is the proof that there are infinitely many prime numbers. We start by assuming that there are *finitely* many primes, so they can be listed as $p_1, p_2, \ldots, p_n$, where p_n is the largest prime number. But then the number $p_1 \times p_2 \times \ldots \times p_n + 1$ must also be prime, because it is not divisible by any of the smaller primes. Furthermore this number is definitely bigger than p_n. So we have contradicted the idea that there was a 'biggest' prime called p_n, and therefore there are infinitely many primes.
- **Proof by mathematical induction:** in mathematical induction, we start with a formula that we *suspect* is true. For example, I might *suspect* from

$\sum_{k=1}^{n} k = 1 + 2 + 3 + \dots + n \quad ?? = \frac{n(n+1)}{2} \; ?$

$n = 10 \quad 1 + 2 + \dots + 10 = \frac{10 \times 11}{2}$ verified.

(True for $n=10$) $\Rightarrow$ (True for $n=11$) (True for $n=x$) $\Rightarrow$ (True for $n=x+1$)

observation that $\sum_{k=1}^{n} k = n(n+1)/2$. I might have tested this formula for many different values of n, but of course I can never test it for *all* values of n. Therefore I need to prove that the formula is *always* true.

The idea of mathematical induction is to say: *suppose* the formula is true for all n up to the value $n = 10$ (say). Can I prove that, *if* it is true for $n = 10$, *then* it will also be true for $n = 11$? And *if* it is true for $n = 11$, then it will also be true for $n = 12$? And so on.

In practice, we usually start lower than $n = 10$. We usually take the very easiest case, $n = 1$, and prove that the formula is true for $n = 1$: LHS $= \sum_{k=1}^{1} k = 1 = 1 \times 2/2 =$ RHS. Then we prove that, *if* the formula is ever true for $n = x$, *then* it will always be true for $n = x + 1$. Because it is true for $n = 1$, it must be true for $n = 2$; and because it is true for $n = 2$, it must be true for $n = 3$; and so on, for all possible n. Thus the formula is proved.

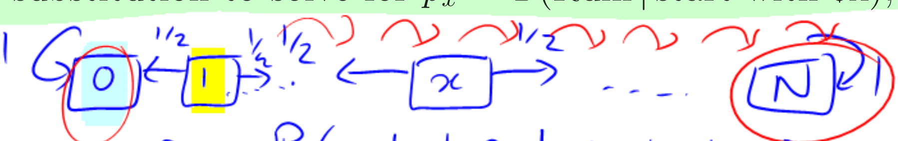
Mathematical induction is therefore a bit like a first-step analysis for proving things: prove that wherever we are now, the next step will always be OK. Then if we were OK at the very begining, we will be OK for ever.

The method of mathematical induction for proving results is very important in the study of Stochastic Processes. This is because a stochastic process builds up one step at a time, and mathematical induction works on the same principle.

→ FSA: $p_0 = 1 \quad p_1 = \frac{1}{2} p_0 + \frac{1}{2} p_2 \Rightarrow \frac{1}{2} + \frac{1}{2} p_2 \Rightarrow p_2 = 2p_1 - 1$

Example: We have already seen examples of inductive-type reasoning in this course. For example, in Chapter 2 for the Gambler's Ruin problem, using the method of repeated substitution to solve for $p_x = \mathbb{P}(\text{Ruin} \mid \text{start with } \$x)$, we discovered that:

- $p_2 = 2p_1 - 1$
- $p_3 = 3p_1 - 2$
- $p_4 = 4p_1 - 3$



$p_x = \mathbb{P}(\text{end at } 0 \mid \text{start at } x)$ for any x.

We deduced that $p_x = x p_1 - (x-1)$ in general.

To prove this properly, we should have used the method of mathematical induction.

5.2 Mathematical Induction by example

This example explains the style and steps needed for a proof by induction.

Question: Prove by induction that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ (∗) for any integer n = 1, 2, 3, ...

Approach: follow the steps below.

(i) First verify that the formula (∗) is true for a *base case*: usually the smallest appropriate value of n (e.g. $n = 0$ or $n = 1$). Here, the smallest possible value of n is $n = 1$, because we can't have $\sum_{k=1}^{0}$.

Base case: $n=1$

First verify that (∗) is true for the first case, $n=1$.

LHS(∗) $= \sum_{k=1}^{1} k = 1$

RHS(∗) $= \frac{1*2}{2} = 1 =$ LHS(∗)

So (∗) is proved for $n=1$.

(ii) Next suppose that formula (⋆) is true for *all values of n up to and including some value x.* (We have already established that this is the case for $x = 1$).
Using the hypothesis that (⋆) is true for all values of n up to and including x, *prove* that it is therefore true for the value $n = x + 1$.

General case: suppose (∗) is true for $n = 1, 2, \ldots, x$ for some x.

So we can assume $\sum_{k=1}^{x} k = \frac{x(x+1)}{2}$. (a)

(a) for "allowed info": we CAN use this in our working, it is not something we are trying to prove.

We need to show that, IF (∗) holds for $n=x$, THEN (∗) will also hold for $n=x+1$.

"Require To Prove"

RTP $\sum_{k=1}^{x+1} k = \frac{(x+1)(x+2)}{2}$. (**)

use *'s for labels where you WANT to prove it, and have not yet done so.

obtained by putting $n=x+1$ everywhere in (*).

$$\text{LHS}(**) = \sum_{k=1}^{x+1} k$$

$$= \sum_{k=1}^{x} k + (x+1) \quad \text{expanding the sum}$$

$$= \frac{x(x+1)}{2} + (x+1) \quad \text{using allowed info (a)}$$

$$= (x+1)\left\{\frac{x}{2} + 1\right\}$$

$$= \frac{(x+1)(x+2)}{2}$$

$$= \text{RHS}(**).$$

So, if (*) holds for $n=x$, we have proved (*) also holds for $n=x+1$.

(iii) Refer back to the base case: if it is true for $n = 1$, then it is true for $n = 1+1 = 2$ by (ii). If it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ by (ii). We could go on forever. This proves that the formula ($\star$) is true for all n.

We proved (*) true for $n=1$,

$\therefore$ (*) is proved for all $n=1,2,3,\ldots$. □

General procedure for proof by induction

The procedure above is quite standard. The inductive proof can be summarized like this:

Question: prove that $f(n) = g(n)$ for all integers $n \geq 1$. $(\star)$

Base case: $n = 1$ RTP Prove that $f(1) = g(1)$ *using*

$$\begin{aligned} \mathit{LHS} &= f(1) \\ &= \vdots \\ &= g(1) = \mathit{RHS}. \end{aligned}$$

General case: *suppose* $(\star)$ *is true for* $n = x$:

so $f(x) = g(x)$. (a) *(allowed info)*

Prove that $(\star)$ *is therefore true for* $n = x + 1$:

RTP $f(x+1) = g(x+1)$. $(\star\star)$ want to prove: label $(\star\star)$

$$\begin{aligned} \mathit{LHS}(\star\star) &= f(x+1) \\ &= \left\{ \begin{array}{c} \text{some expression breaking down } f(x+1) \\ \text{into } f(x) \text{ and an extra term in } x+1 \end{array} \right\} \\ &= \{ \text{substitute } f(x) = g(x) \text{ in the line above} \} \quad \text{by allowed (a)} \\ &= \{\text{do some working}\} \\ &= g(x+1) \\ &= \mathit{RHS}(\star\star). \end{aligned}$$

Conclude: $(\star)$ *is proved for* $n = 1$, *so it is proved for* $n = 2$, $n = 3$, $n = 4, \ldots$

$(\star)$ *is therefore proved for all integers* $n \geq 1$. $\square$

5.3 Some harder examples of mathematical induction

Induction problems in stochastic processes are often trickier than usual. Here are some possibilities:

- Backwards induction: start with base case $n = N$ and go backwards, instead of starting at base case $n = 1$ and going forwards.
- Two-step induction, where the proof for $n = x + 1$ relies not only on the formula being true for $n = x$, but also on it being true for $n = x - 1$.

The first example below is hard probably because it is too easy. The second example is an example of a two-step induction. $p_{x+1} = \frac{1}{\alpha} p_x$ $\quad p_0 = 1$, $p_1 = \frac{1}{\alpha}$, $p_2 = \frac{1}{\alpha^2}$

Example 1: Suppose that $p_0 = 1$ and $p_x = \alpha p_{x+1}$ for all $x = 1, 2, \ldots$. Prove by mathematical induction that $p_n = 1/\alpha^n$ for $n = 0, 1, 2, \ldots$.

RTP $p_n = \frac{1}{\alpha^n}$ for $n = 0, 1, 2, \ldots$ (*)

Information given: $p_{x+1} = \frac{1}{\alpha} p_x$ (G1)

$p_0 = 1$ (G2)

"G" for "given info"

Base case: $n = 0$.

LHS (*) $= p_0 = 1$ by given (G2).

RHS (*) $= \frac{1}{\alpha^0} = 1 = $ LHS (*).

So (*) is proved for case $n = 0$.

General case: Suppose (*) is true for $n = x$.

So we can assume $p_x = \frac{1}{\alpha^x}$ (a) (allowed info)

RTP (*) true for $n = x+1$:

i.e. RTP $p_{x+1} = \frac{1}{\alpha^{x+1}}$ (**)

Test: arrive before 11am

BLT 204

Bring Paper, Calculator, Stapler if poss.

Revise: A1 A2 || **

// Write down FSA notation & equations first

solve all at end.

LHS(**) $= p_{x+1}$

$= \frac{1}{\alpha} p_x$ by given (G1)

$= \frac{1}{\alpha} * \frac{1}{\alpha^x}$ by allowed (a)

$= \frac{1}{\alpha^{x+1}}$

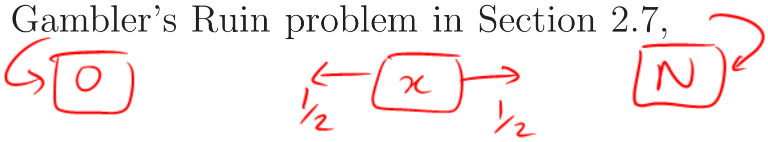
= RHS(**).

So if (*) is proved for $n=x$, we have proved (*) for $n=x+1$.

We proved (*) for base case $n=0$,

∴ (*) is proved for all $n = 0, 1, 2, 3, \ldots$ ☐

Example 2: ***Gambler's Ruin.*** In the Gambler's Ruin problem in Section 2.7, we have the following situation:



- $p_x = \mathbb{P}(\text{Ruin} \mid \text{start with \$x})$;
- We know from first-step analysis that $p_{x+1} = 2p_x - p_{x-1}$ (G_1)
- We know from common sense that $p_0 = 1$ (G_2)
- By direct substitution into (G_1), we obtain:

$$\begin{aligned} p_2 &= 2p_1 - 1 \\ p_3 &= 3p_1 - 2 \end{aligned}$$

- We develop a suspicion that for all $x = 1, 2, 3, \ldots,$

$$p_x = xp_1 - (x-1) \qquad (\star)$$

- We wish to prove $(\star)$ by mathematical induction.

For this example, our given information, in (G1), expresses p_{x+1} in terms of both p_x and p_{x-1}, so we need TWO base cases. Use $x=1$ and $x=2$.

RTP $p_x = x p_1 - (x-1)$ (*) for $x = 1, 2, \ldots, N$.

Base case $x=1$:

LHS(*) $= p_1$

RHS(*) $= 1 * p_1 - 0 = p_1 =$ LHS(*).

So (*) is proved for case $x=1$.

Base case $x=2$:

LHS(*) $= p_2 = 2p_1 - 1$ by given info (G1) and (G2)

RHS(*) $= 2p_1 - 1 =$ LHS(*)

So (*) is proved for case $x=2$.

General case: suppose (*) is true for all $x = 1, 2, \ldots, k$.

So we are allowed:

$(x=k)$ $\quad p_k = k p_1 - (k-1)$ (a1)

$(x=k-1)$ $\quad p_{k-1} = (k-1) p_1 - (k-2)$ (a2)

} allowed info.

RTP (*) is then true for $x = k+1$:

i.e. RTP $p_{k+1} = (k+1) p_1 - k$ (**)

LHS(**) $= p_{k+1}$

$= 2p_k - p_{k-1}$ by given info (G1)

$= 2\{k p_1 - (k-1)\} - \{(k-1) p_1 - (k-2)\}$

by allowed info (a1) and (a2)

$= p_1 \{2k - (k-1)\} - \{2(k-1) - (k-2)\}$

$= (k+1) p_1 - k$

$=$ RHS(**).

So if (*) is true for $x = 1, 2, \ldots, k$ ($k \geq 2$), it is also true for $x = k+1$. We proved (*) for $x=1$ and $x=2$,

So (*) is proved for all $x = 1, 2, \ldots, N$. □