Chapter 5: Mathematical Induction

So far in this course, we have seen some techniques for dealing with stochastic processes: first-step analysis for hitting probabilities (Chapter 2), and first-step analysis for expected reaching times (Chapter 3). We now look at another tool that is often useful for exploring properties of stochastic processes:

5.1 Proving things in mathematics

There are many different ways of constructing a formal proof in mathematics. Some examples are:

- **Proof by counterexample:** a proposition is proved to be *not generally true* because a *particular example* is found for which it is not true.
- **Proof by contradiction:** this can be used either to prove a proposition is true or to prove that it is false. To prove that the proposition is *true* (say), we start by *assuming that it is false.* We then explore the consequences of this assumption until we reach a contradiction, e.g. $0 = 1$. Therefore something must have gone wrong, and the only thing we weren't sure about was our initial assumption that the proposition is false — so our initial assumption must be wrong and the proposition is proved true.

  A famous proof of this sort is the proof that there are infinitely many prime numbers. We start by assuming that there are *finitely* many primes, so they can be listed as $p_1, p_2, \ldots, p_n$, where p_n is the largest prime number. But then the number $p_1 \times p_2 \times \ldots \times p_n + 1$ must also be prime, because it is not divisible by any of the smaller primes. Furthermore this number is definitely bigger than p_n. So we have contradicted the idea that there was a 'biggest' prime called p_n, and therefore there are infinitely many primes.
- **Proof by mathematical induction:** in mathematical induction, we start with a formula that we *suspect* is true. For example, I might *suspect* from

observation that $\sum_{k=1}^{n} k = n(n+1)/2$. I might have tested this formula for many different values of n, but of course I can never test it for *all* values of n. Therefore I need to prove that the formula is *always* true.

The idea of mathematical induction is to say: *suppose* the formula is true for all n up to the value $n = 10$ (say). Can I prove that, *if* it is true for $n = 10$, *then* it will also be true for $n = 11$? And *if* it is true for $n = 11$, then it will also be true for $n = 12$? And so on.

In practice, we usually start lower than $n = 10$. We usually take the very easiest case, $n = 1$, and prove that the formula is true for $n = 1$: LHS $= \sum_{k=1}^{1} k = 1 = 1 \times 2/2 =$ RHS. Then we prove that, *if* the formula is ever true for $n = x$, *then* it will always be true for $n = x + 1$. Because it is true for $n = 1$, it must be true for $n = 2$; and because it is true for $n = 2$, it must be true for $n = 3$; and so on, for all possible n. Thus the formula is proved.

Mathematical induction is therefore a bit like a

The method of mathematical induction for proving results is very important in the study of Stochastic Processes. This is because a stochastic process builds up one step at a time, and mathematical induction works on the same principle.

Example: We have already seen examples of inductive-type reasoning in this course. For example, in Chapter 2 for the Gambler's Ruin problem, using the method of repeated substitution to solve for $p_x = \mathbb{P}(\text{Ruin} \,|\, \text{start with \$x})$, we discovered that:

- $p_2 = 2p_1 - 1$
- $p_3 = 3p_1 - 2$
- $p_4 = 4p_1 - 3$

We deduced that

To prove this properly, we should have used the method of mathematical induction.

5.2 Mathematical Induction by example

This example explains the style and steps needed for a proof by induction.

Question: Prove by induction that $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for any integer n.

Approach: follow the steps below.

(i) First verify that the formula is true for a *base case*: usually the smallest appropriate value of n (e.g. $n = 0$ or $n = 1$). Here, the smallest possible value of n is $n = 1$, because we can't have $\sum_{k=1}^{0}$.

(ii) Next suppose that formula ($\star$) is true for *all values of n up to and including some value x.* (We have already established that this is the case for $x = 1$).

Using the hypothesis that ($\star$) is true for all values of n up to and including x, *prove* that it is therefore true for the value $n = x + 1$.

(iii) Refer back to the base case: if it is true for $n = 1$, then it is true for $n = 1+1 = 2$ by (ii). If it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ by (ii). We could go on forever. This proves that the formula $(\star)$ is true for all n.

General procedure for proof by induction

The procedure above is quite standard. The inductive proof can be summarized like this:

Question: prove that $f(n) = g(n)$ for all integers $n \geq 1$. $(\star)$

Base case: $n = 1$. *Prove that* $f(1) = g(1)$ *using*

$$\begin{aligned} LHS &= f(1) \\ &= \vdots \\ &= g(1) = RHS. \end{aligned}$$

General case: *suppose* $(\star)$ *is true for* $n = x$:

$$\text{so} \qquad f(x) = g(x). \qquad (a) \quad \text{(allowed info)}$$

Prove that $(\star)$ *is therefore true for* $n = x + 1$:

$$RTP \qquad f(x+1) = g(x+1). \qquad (\star\star)$$

$$\begin{aligned} LHS(\star\star) &= f(x+1) \\ &= \left\{ \begin{array}{c} \text{some expression breaking down } f(x+1) \\ \text{into } f(x) \text{ and an extra term in } x+1 \end{array} \right\} \\ &= \left\{ \text{substitute } f(x) = g(x) \text{ in the line above} \right\} \qquad \text{by allowed (a)} \\ &= \{\text{do some working}\} \\ &= g(x+1) \\ &= RHS(\star\star). \end{aligned}$$

Conclude: $(\star)$ *is proved for* $n = 1$*, so it is proved for* $n = 2$*,* $n = 3$*,* $n = 4, \ldots$

$(\star)$ *is therefore proved for all integers* $n \geq 1$. $\square$

5.3 Some harder examples of mathematical induction

Induction problems in stochastic processes are often trickier than usual. Here are some possibilities:

- Backwards induction: start with base case $n = N$ and go backwards, instead of starting at base case $n = 1$ and going forwards.
- Two-step induction, where the proof for $n = x + 1$ relies not only on the formula being true for $n = x$, but also on it being true for $n = x - 1$.

The first example below is hard probably because it is too easy. The second example is an example of a two-step induction.

Example 1: Suppose that $p_0 = 1$ and $p_x = \alpha p_{x+1}$ for all $x = 1, 2, \ldots$. Prove by mathematical induction that $p_n = 1/\alpha^n$ for $n = 0, 1, 2, \ldots$.

Example 2: Gambler's Ruin. In the Gambler's Ruin problem in Section 2.7, we have the following situation:

- $p_x = \mathbb{P}(\text{Ruin} \,|\, \text{start with \$x})$;
- We know from first-step analysis that $p_{x+1} = 2p_x - p_{x-1}$ (G_1)
- We know from common sense that $p_0 = 1$ (G_2)
- By direct substitution into (G_1), we obtain:

$$\begin{aligned} p_2 &= 2p_1 - 1 \\ p_3 &= 3p_1 - 2 \end{aligned}$$

- We develop a suspicion that for all $x = 1, 2, 3, \ldots,$

$$p_x = xp_1 - (x - 1) \qquad (\star)$$

- We wish to prove $(\star)$ by mathematical induction.

For this example,