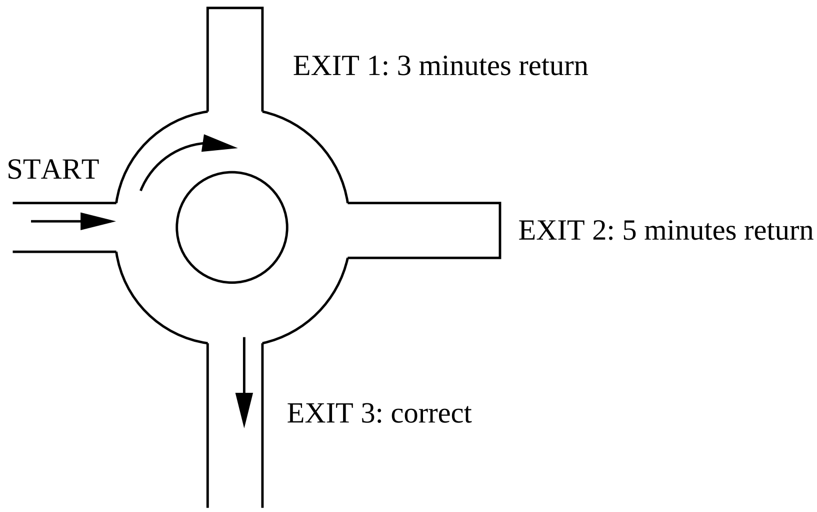
Answer **ALL THREE QUESTIONS**. Marks are shown for each question.
A **FORMULA SHEET** is overleaf.

1. A lecturer is driving around Auckland. She comes to a roundabout with three exits. Exit 1 is a dead end, and if she takes this exit she will arrive back at the roundabout after 3 minutes. Exit 2 is also a dead end, and will lead her back to the roundabout after 5 minutes. Exit 3 is the correct exit.

The lecturer has no idea which exit is correct. She chooses one at random. If she makes the wrong choice and ends up back at the roundabout again, she will make another random choice. She never chooses the same exit twice: for example, if she chooses Exit 1 the first time, she will not choose Exit 1 again.



(a) Let T be the total length of time before the lecturer chooses the correct exit, Exit 3. Find $\mathbb{E}(T)$. (8)

(b) Now suppose that the forgetful lecturer **can** choose an exit that she has chosen before. Thus she can choose Exit 1 on the first attempt, Exit 2 on the second attempt, Exit 1 again on the third attempt, and so on. However, she will never make the same choice twice in succession. When she emerges from Exit 1, for example, she can choose either Exit 2 or Exit 3 with equal probability, but not Exit 1 again. (She always chooses one of Exit 1, 2, or 3.)
Ignoring the time taken while driving on the roundabout itself, define

$$T_1 = \text{time taken to leave roundabout, starting from Exit 1;}$$

$$T_2 = \text{time taken to leave roundabout, starting from Exit 2.}$$

Show that $\mathbb{E}(T_1) = 3 + \frac{1}{2}\mathbb{E}(T_2)$, and find a similar expression for $\mathbb{E}(T_2)$. (4)

(c) Solve the expressions in part (b) to find $\mathbb{E}(T_1)$ and $\mathbb{E}(T_2)$. (4)

(d) Using your answer to part (c), find $\mathbb{E}(T)$, where T is the **total** length of time before the lecturer leaves the roundabout, starting from the point marked START on the diagram. (2)

2. A friendly ghost has difficulty in passing through walls.

Usually, he gets through the wall all right and remains himself (probability 0.7).

Sometimes, he accidentally splits into two and makes two copies of himself, one on each side of the wall (probability 0.2).

Occasionally, he gets it totally wrong and disappears altogether (probability 0.1).

Whenever the ghost splits into two, all copies of the ghost behave independently, with the same probabilities as above. Every ghost tries to pass through the wall every five minutes to keep practising (although they never improve).

Let $\{Z_0 = 1, Z_1, Z_2, \ldots\}$ be a branching process describing the number of ghosts after each attempt at passing through the wall. Thus $Z_0 = 1$ is the ghost at the start; Z_1 is the number of ghosts after the first attempt, Z_2 is the number after the second attempt, and so on.

(a) Let Y denote the family size distribution of the branching process. Write down the probability function of Y in the following format:

y	0	1	$\ldots$
$\mathbb{P}(Y = y)$			

(1)

(b) Find the probability generating function of Y, $G(s) = \mathbb{E}(s^Y)$. (2)

(c) Let $G_2(s)$ be the probability generating function of Z_2, the number of ghosts after two attempts at passing through walls. Find $G_2(s)$. (You do **not** need to simplify the expression.) (3)

(d) Using the PGF $G_2(s)$, find $\mathbb{P}(Z_2 = 0)$. (2)

(e) Using the PGF $G_2(s)$, find $\mathbb{E}(Z_2)$. (4)

FORMULAE

Probability Generating Function: $G_X(s) = \mathbb{E}(s^X)$.

$\mathbb{P}(X = 0) = G_X(0)$. Mean: $\mathbb{E}(X) = G'_X(1)$.