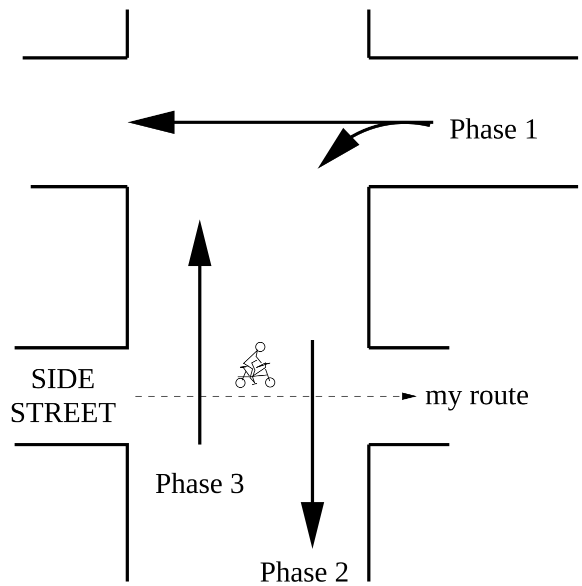
Answer **ALL THREE QUESTIONS**. Marks are shown for each question.
A **FORMULA SHEET** is overleaf.

1. Every day as I cycle to work, I have to cross a main road from a side street. The traffic on the main road is controlled by traffic lights. There are three phases of the traffic lights, where each phase allows a different stream of traffic to pass. Every phase lasts exactly 20 seconds. When the phase changes, I might be able to cross quickly; otherwise, I have to wait 20 seconds before the next phase. The details are as follows:

1. Phase 1: cross immediately with probability 0.8; otherwise wait 20 seconds and go to Phase 2.
2. Phase 2: cross immediately with probability 0.4; otherwise wait 20 seconds and go to Phase 3.
3. Phase 3: cross immediately with probability 0.1; otherwise wait 20 seconds and go to Phase 1.



(Note: the traffic diagram above is provided for interest, but all the information you need for the question is provided in points 1, 2, and 3 above.)

Define

$$T_1 = \text{time in seconds from the } \textbf{start} \text{ of Phase 1 until I can cross;}$$
$$T_2 = \text{time in seconds from the } \textbf{start} \text{ of Phase 2 until I can cross;}$$
$$T_3 = \text{time in seconds from the } \textbf{start} \text{ of Phase 3 until I can cross.}$$

(a) Show that $\mathbb{E}(T_1) = 4 + 0.2\,\mathbb{E}(T_2)$, and derive similar expressions for $\mathbb{E}(T_2)$ and $\mathbb{E}(T_3)$. (6)

(b) By solving the expressions from part (a), show that $\mathbb{E}(T_1) = 9.6$ seconds, and find $\mathbb{E}(T_2)$ and $\mathbb{E}(T_3)$. (6)

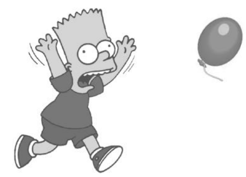
(c) When I first arrive at the intersection, the traffic lights are equally likely to be at any point in their cycle. I therefore have to wait time $W \sim \text{Uniform}(0, 20)$ seconds before the next phase starts, and then it is equally likely to be Phase 1, Phase 2, or Phase 3.

Let T be the total time I have to wait, from the moment I arrive at the intersection, to the moment I can cross the road. Find $\mathbb{E}(T)$. (3)

2. Bart is trying to catch a runaway balloon.

He runs after it for $R \sim$ Exponential(0.1) seconds, then he can try to grab it. With probability 0.2, he is successful. With probability 0.8, he falls over and has to start again from the beginning.

Let T be the total time that Bart takes to catch the balloon. Find $\mathbb{E}(T)$. (4)



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3. The Loch Ness Monster is a famous type of dinosaur that lives in Scotland.

The breeding cycle of the Loch Ness Monster is as follows. Every monster has either 3 offspring (probability 0.7), or no offspring at all (probability 0.3). All monsters act independently of each other. (There are no males and females.)

Nessie is a Loch Ness Monster. Let $\{Z_0 = 1, Z_1, Z_2, \ldots\}$ be a branching process describing direct descendants of Nessie: so Z_1 is the number of Nessie's children, Z_2 is the number of her grandchildren, and so on.

(a) Let Y denote the family size distribution of the branching process. Write down the probability function of Y in the following format: (1)

y	0	1	...
$\mathbb{P}(Y = y)$			

(b) Find the probability generating function of Y, $G(s) = \mathbb{E}(s^Y)$. (2)

(c) Let $G_2(s)$ be the probability generating function of Z_2, the number of Nessie's grandchildren. Find $G_2(s)$. (You do **not** need to simplify the expression.) (3)

(d) Using the PGF $G_2(s)$, find $\mathbb{P}(Z_2 = 0)$. (2)

(e) Using the PGF $G_2(s)$, find $\mathbb{E}(Z_2)$. (3)

FORMULAE

Uniform Distribution: $X \sim \text{Uniform}(a, b)$. Mean: $\mathbb{E}(X) = \dfrac{a+b}{2}$.

Exponential Distribution: $X \sim \text{Exponential}(\lambda)$. Mean: $\mathbb{E}(X) = \dfrac{1}{\lambda}$.

Probability Generating Function: $G_X(s) = \mathbb{E}(s^X)$.

$\mathbb{P}(X = 0) = G_X(0)$. Mean: $\mathbb{E}(X) = G'_X(1)$.