Answer **ALL THREE QUESTIONS**. Detach the **ANSWER SHEET** on the back, and use it for Question 1(a) and any additional questions as you choose.

Marks are shown for each question. The following **FORMULAE** may be helpful.

# FORMULAE

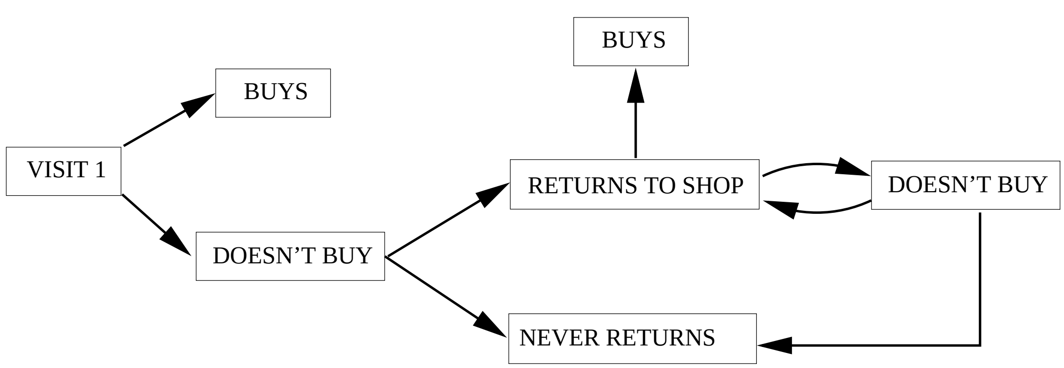
**Probability Generating Function:** $G_X(s) = \mathbb{E}(s^X)$.

$\mathbb{P}(X = 0) = G_X(0)$. Mean: $\mathbb{E}(X) = G'_X(1)$.

**1.** Mr Forte runs a piano shop. He observes the following behaviour in his customers. On their first visit to the shop, the customer will buy a piano with probability 0.1. With probability 0.9, they do not buy immediately, but go away promising to think about it.

If the customer does *not* buy on the first visit, there is a 0.2 chance that they will return to the shop, and a 0.8 chance that they will never return. From the second visit onwards, there is a 0.5 chance that the customer will buy on that visit, and a 0.5 chance that they will not buy and go away to think about it again. If the customer does not buy and goes away to think again, there is a 0.5 chance that they will return for a new visit to the shop, and a 0.5 chance that they will never return. The customer's behaviour for all subsequent visits is identical to that on the second visit.

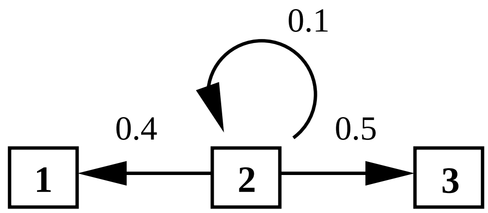
(a) The diagram below is reproduced on the answer sheet. Add the relevant probability to each arrow on the diagram. (2)




(b) Find the probability that a customer eventually buys a piano from the shop. (4)

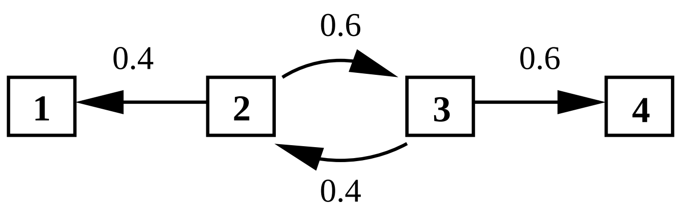
(c) Let $N$ be the total number of times the customer visits the shop, including the first visit. Find $\mathbb{E}(N)$. (4)

**2.**(a) Consider the system represented by the diagram below. Arrows denote possible moves, and the numbers over the arrows represent the probabilities of these moves. The possible states are 1, 2, and 3, as marked in the boxes.




Let $T$ be the number of steps taken, starting from State 2, to reach either State 1 or State 3. Let $H(s) = \mathbb{E}(s^T)$ be the probability generating function of $T$. Find $H(s)$, and state its radius of convergence. (4)

(b) Now consider the system represented by the new diagram. This time, let $T$ be the number of steps taken, starting from State 2, to reach either State 1 or State 4. Let $H(s) = \mathbb{E}(s^T)$ be the probability generating function of $T$. Find $H(s)$. (6)

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**3.** A computer virus is passed from one machine to another. Whenever it infects a machine, it is discovered and removed with probability 0.8. If it is not removed (probability 0.2), it will wait until the next Monday morning, then infect 10 new machines, and cause the original machine to explode. The process continues in exactly the same way for each of the new infected machines. Assume that all machines are independent of each other.

Let $\{Z_0 = 1, Z_1, Z_2, \ldots\}$ be a branching process describing the progress of the virus. A single machine is infected at the start, so $Z_0 = 1$. $Z_1$ is the number of machines infected directly from the initial machine on the first Monday morning. $Z_2$ is the number of machines infected from these new machines on the second Monday morning; and so on.

(a) Let $Y$ denote the family size distribution of the branching process. Write down the probability function of $Y$ in the following format: (1)

| $y$ | 0 | $\ldots$ |
|---|---|---|
| $\mathbb{P}(Y = y)$ | | |

(b) Find the probability generating function of $Y$, $G(s) = \mathbb{E}(s^Y)$. (1)

(c) Let $G_2(s)$ be the probability generating function of $Z_2$. Find $G_2(s)$. (You do **not** need to simplify the expression.) (3)

(d) Using the PGF $G_2(s)$, find $\mathbb{P}(Z_2 = 0)$. (2)

(e) Using the PGF $G_2(s)$, find $\mathbb{E}(Z_2)$. (3)

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# Answer Sheet

**Name:**

**ID:**

Detach this sheet and hand in with any other working. You may use both sides of this sheet if you choose.

**Question 1(a).** Add the probabilities to the diagram below.