Answer **ALL FOUR QUESTIONS**. Marks are shown for each question. The following **FORMULAE** may be helpful.

FORMULAE

Probability Generating Function: $G_X(s) = \mathbb{E}(s^X)$.

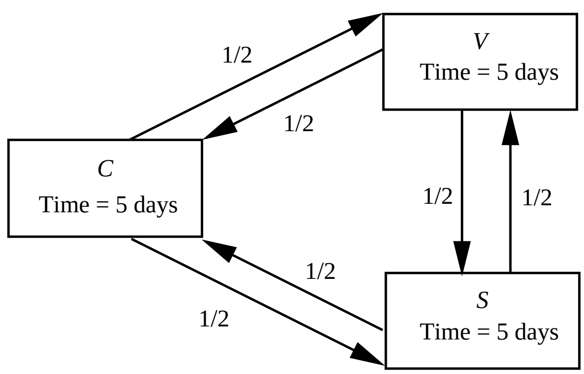
$\mathbb{P}(X = 0) = G_X(0)$. Mean: $\mathbb{E}(X) = G'_X(1)$.

1. The Anarchists' Society has a very disorganised committe. There are three people on the committee: the Chair (C), the Vice (V), and the Secretary (S). When a committee member receives a letter for the society, they wait 5 days and then pass it on to one of the other two committee members at random.

   Define the following states:

$$\begin{aligned} C &= \{\text{letter is with the Chair}\} \\ V &= \{\text{letter is with the Vice}\} \\ S &= \{\text{letter is with the Secretary}\} \end{aligned}$$

   The transition diagram is shown below.



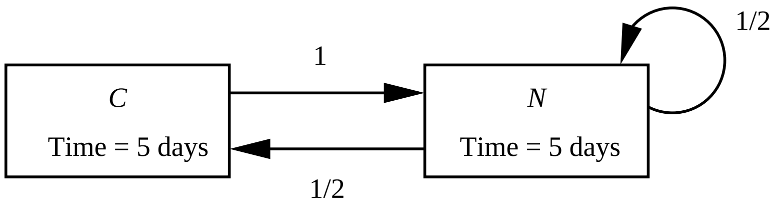
   Let E_C be the expected number of days taken from first *entering* state C to next *entering* state C.

   Let E_S and E_V be the expected number of days to enter **state C**, starting respectively from states S and V.

   Show that $E_C = 5 + \frac{1}{2}E_V + \frac{1}{2}E_S$, and find expressions for E_V and E_S. Solve the expressions to find E_C, E_V, and E_S. (6)

[continued...]

2. The system in Question 1 can be redrawn as two states: C and N, where N represents any committee member that is Not C. The diagram is below.



Define:

$$\begin{aligned} T &= \text{time in days from entering state } C \text{ to next entering } \textbf{state } \boldsymbol{C} \\ W &= \text{time in days from entering state } N \text{ to next entering } \textbf{state } \boldsymbol{C} \end{aligned}$$

Also define the probability generating functions of T and W as:

$$\begin{aligned} G(s) &= \mathbb{E}(s^T) \\ H(s) &= \mathbb{E}(s^W) \end{aligned}$$

(a) Show that

$$H(s) = \frac{s^5}{2 - s^5}\,.$$

(4)

(b) Using the expression in (a), find $G(s)$. (2)

(c) State the **name** of the quantity in Question 1 that corresponds to $G'(1)$. No calculation is required. (1)

[continued...]

3. The Anarchists' Society decides to improve its procedures for handling letters. Letters are passed round the committee as before in Question 1. However, under the new procedures, when a committee member sees the letter for the second time, they throw it in the bin.

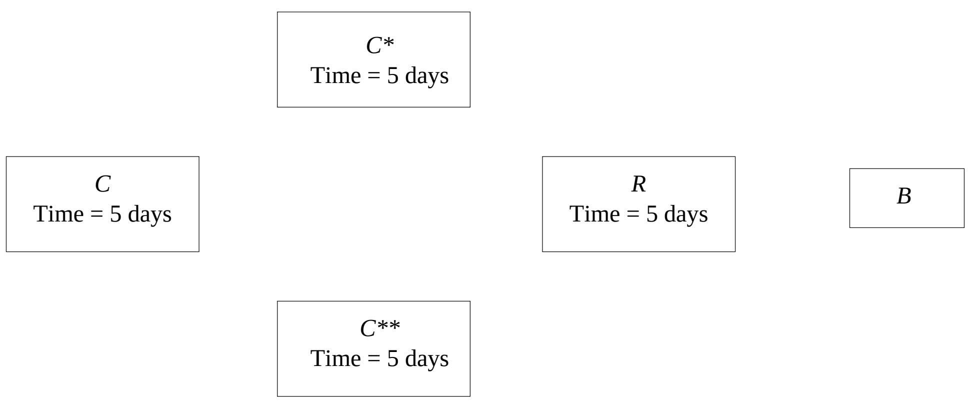
The process is as follows.

- Every time the letter reaches a person, it takes 5 days before it either reaches the next person or is binned.
- Each person bins the letter if and only if they have already seen it before. Otherwise, they pass it on to either of the other two people with equal probability.

Suppose a letter starts with the Chair (state C). Define the following states:

$$
\begin{aligned}
C &= \{\text{letter has been seen by the Chair only}\} \\
C* &= \{\text{letter has been seen by the Chair and ONE other person}\} \\
C{**} &= \{\text{letter has been seen by the Chair and TWO other people}\} \\
R &= \{\text{letter has reached somebody for the second time}\} \\
B &= \{\text{letter is binned}\}
\end{aligned}
$$

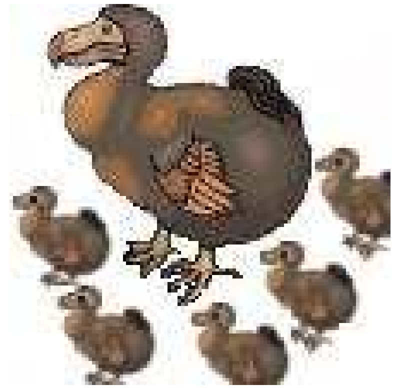
(a) The diagram below shows the path of a letter that is sent initially to the Chair. The times in each box refer to the amount of time spent in that box only. Copy the diagram, and add the appropriate arrows and probabilities. (4)



(b) Find the expected time taken from entering state C (letter first reaches the Chair) to reaching state B (letter is binned). (3)

[continued...]

4. Dudley the Dodo decides to reproduce according to a branching process. He and all his descendants will have either no offspring, with probability 0.8, or 5 offspring, with probability 0.2.



Let $\{Z_0 = 1, Z_1, Z_2, \ldots\}$ be the branching process started by Dudley. At the beginning, the population consists of Dudley alone, so $Z_0 = 1$. Z_1 represents the number of Dudley's offspring; Z_2 is the number of his grand-offspring, and so on.

(a) Let Y denote the family size distribution of the branching process. Write down the probability function of Y in the following format: (1)

y	0	$\ldots$
$\mathbb{P}(Y = y)$		

(b) Find the probability generating function of Y, $G(s) = \mathbb{E}(s^Y)$. (1)

(c) Let $G_2(s)$ be the probability generating function of Z_2. Find $G_2(s)$. (You do **not** need to simplify the expression.) (3)

(d) Using the PGF $G_2(s)$, find $\mathbb{P}(Z_2 = 0)$. (2)

(e) Using the PGF $G_2(s)$, find $\mathbb{E}(Z_2)$. (3)

Total: 30