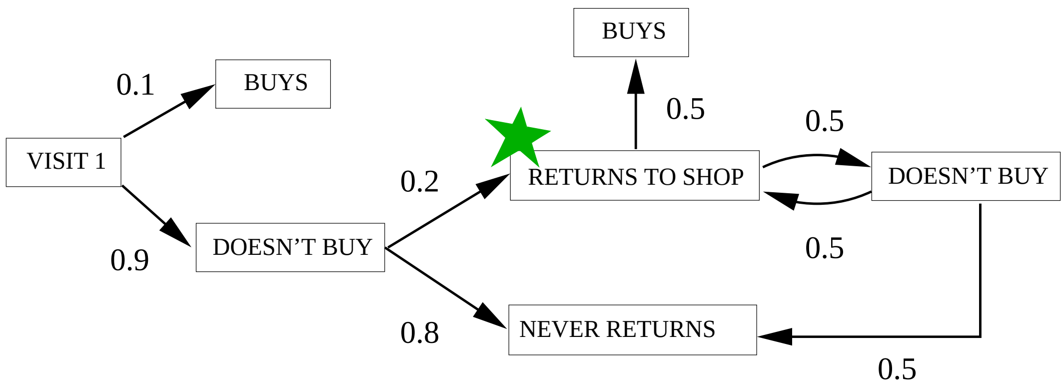
1.(a) The probabilities are marked on the diagram.



(b) Let p be the probability that a customer eventually buys a piano.
Let r be the probability that a customer eventually buys a piano, starting from *entry* to the box marked $\star$ on the diagram above.
Then

$$p = 0.1 + 0.9 \times 0.2 \times r. \qquad (1)$$

Now

$$\begin{aligned} r &= 0.5 + (0.5)^2 r \\ \Rightarrow \quad r(1-(0.5)^2) &= 0.5 \\ \Rightarrow \quad r &= \frac{0.5}{1-0.25} = \frac{2}{3}. \end{aligned}$$

Thus, from (1),

$$\mathbb{P}(\text{buys}) = p = 0.1 + 0.9 \times 0.2 \times r = 0.22.$$

(c) Let N be the number of times that a customer visits the shop, including the first time.
Let M be the number of times that a customer visits the shop, starting from *entry* to the box marked $\star$ on the diagram above.
Then

$$\mathbb{E}(N) = 1 + 0.9 \times 0.2 \times \mathbb{E}(M). \qquad (2)$$

Now

$$\begin{aligned} \mathbb{E}(M) &= 1 + (0.5)^2 \mathbb{E}(M) \\ \Rightarrow \quad \mathbb{E}(M)(1-(0.5)^2) &= 1 \\ \Rightarrow \quad \mathbb{E}(M) &= \frac{1}{1-0.25} = \frac{4}{3}. \end{aligned}$$

Thus, from (2),

$$\mathbb{E}(\text{number of visits}) = \mathbb{E}(N) = 1 + 0.9 \times 0.2 \times \mathbb{E}(M) = 1.24.$$

2.(a) Let T be the number of steps to reach either state 1 or 3. We can write:

$$T = \begin{cases} 1 & \text{with probability } 0.9, \\ 1 + T_1 & \text{with probability } 0.1, \text{ where } T_1 \sim T. \end{cases}$$

Thus if $H(s) = \mathbb{E}(s^T)$, then

$$\begin{aligned} H(s) &= 0.9\mathbb{E}(s^1) + 0.1\mathbb{E}(s^{1+T_1}) \\ &= 0.9s + 0.1s\mathbb{E}(s^{T_1}) \\ &= 0.9s + 0.1sH(s) \qquad \text{because } T_1 \sim T, \text{ so } \mathbb{E}(s^{T_1}) = \mathbb{E}(s^T) = H(s). \end{aligned}$$

Thus

$$\begin{aligned} H(s)(1 - 0.1s) &= 0.9s \\ \Rightarrow \quad H(s) &= \frac{0.9s}{1 - 0.1s}. \end{aligned}$$

The radius of convergence is 10, because $(1 - 0.1s) = 0$ when $s = 10$.

(b) Let T be the number of steps to reach either state 1 or 4, starting from state 2.
Let W be the number of steps to reach either state 1 or 4, starting from state 3.
Let $H(s) = \mathbb{E}(s^T)$, and let $G(s) = \mathbb{E}(s^W)$.
Then:

$$T = \begin{cases} 1 & \text{with probability } 0.4, \\ 1 + W & \text{with probability } 0.6. \end{cases}$$

This statement for T gives:

$$H(s) = \mathbb{E}(s^T) = 0.4\mathbb{E}(s^1) + 0.6\mathbb{E}(s^{1+W}) = 0.4s + 0.6sG(s). \qquad (1)$$

Similarly,

$$W = \begin{cases} 1 & \text{with probability } 0.6, \\ 1 + T & \text{with probability } 0.4. \end{cases}$$

This statement for W gives:

$$G(s) = \mathbb{E}(s^W) = 0.6\mathbb{E}(s^1) + 0.4\mathbb{E}(s^{1+T}) = 0.6s + 0.4sH(s). \qquad (2)$$

Thus (1) gives:

$$\begin{aligned} H(s) &= 0.4s + 0.6sG(s) \\ &= 0.4s + 0.6s\Big(0.6s + 0.4sH(s)\Big) \qquad \text{from (2)} \\ \Rightarrow \quad H(s)\Big(1 - 0.6s \times 0.4s\Big) &= 0.4s + 0.6s \times 0.6s \\ \Rightarrow \quad H(s) &= \frac{0.4s + 0.36s^2}{1 - 0.24s^2}. \end{aligned}$$

3.(a) The probability function is:

y	0	10
$\mathbb{P}(Y = y)$	0.8	0.2

(b)

$$G(s) = 0.8 + 0.2s^{10}.$$

(c)

$$\begin{aligned} G_2(s) &= G\Big(G(s)\Big) \\ &= 0.8 + 0.2\Big(G(s)\Big)^{10} \\ &= 0.8 + 0.2\Big(0.8 + 0.2s^{10}\Big)^{10}. \end{aligned}$$

(d)

$$\mathbb{P}(Z_2 = 0) = G_2(0) = 0.8 + 0.2\Big(0.8\Big)^{10} = 0.821.$$

(e) We need $\mathbb{E}(Z_2) = G_2'(1)$. Now

$$G_2'(s) = 0.2 \times 10\Big(0.8 + 0.2s^{10}\Big)^9 \times 2s^9,$$

so

$$G_2'(1) = 0.2 \times 10\Big(0.8 + 0.2\Big)^9 \times 2 = 4.$$

Thus

$$\mathbb{E}(Z_2) = G_2'(1) = 4.$$