1. Note first that if we leave state V and go to state C, we have re-entered state C and the time requirement at this point is 0. Similarly, the time requirement from leaving state S and going to state C is 0.

   The expressions are therefore:

$$\begin{aligned} E_C &= 5 + \frac{1}{2}E_V + \frac{1}{2}E_S \\ E_V &= 5 + \frac{1}{2} \times 0 + \frac{1}{2}E_S \\ E_S &= 5 + \frac{1}{2} \times 0 + \frac{1}{2}E_V \end{aligned}$$

   (Each state requires an initial time of 5 days, followed by a choice which takes it to each of the other two states with probability $\frac{1}{2}$ each.)

   Overall, the three equations are:

$$E_C = 5 + \frac{1}{2}E_V + \frac{1}{2}E_S \qquad (1)$$

$$E_V = 5 + \frac{1}{2}E_S \qquad (2)$$

$$E_S = 5 + \frac{1}{2}E_V \qquad (3)$$

   Solving:

$$\begin{aligned} \text{(3) in (2):} \qquad E_V &= 5 + \frac{1}{2}\left(5 + \frac{1}{2}E_V\right) \\ \Rightarrow \qquad \frac{3}{4}E_V &= 5 + \frac{5}{2} \\ \Rightarrow \qquad E_V &= \frac{4 \times 5 + 2 \times 5}{3} \\ E_V &= 10 \text{ days.} \end{aligned}$$

   By symmetry (or substitution in (3)),

$$E_S = 10 \text{ days.}$$

   Substituting in (1):

$$E_C = 5 + \frac{1}{2} \times 10 + \frac{1}{2} \times 10 = 15 \text{ days.}$$

(6)

2.(a) Let W be the time in days from entering state N to returning to state C.

$$W = \begin{cases} 5 & \text{with probability } \frac{1}{2} \quad \text{(returns straight to } C\text{)}, \\ 5 + W_1 & \text{with probability } \frac{1}{2} \quad \text{(stays in } N\text{)}, \end{cases}$$

where $W_1 \sim W$.

Thus

$$\begin{aligned} H(s) \;=\; \mathbb{E}(s^W) &= \frac{1}{2}\mathbb{E}(s^5) + \frac{1}{2}\mathbb{E}(s^{5+W_1}) \\ &= \frac{s^5}{2} + \frac{s^5}{2}H(s) \qquad \text{because } W_1 \sim W \text{ so } \mathbb{E}(s^{W_1}) = H(s) \\ \Rightarrow \quad H(s)\left(1 - \frac{s^5}{2}\right) &= \frac{s^5}{2} \\ \Rightarrow \quad H(s) &= \frac{s^5}{2 - s^5}, \qquad \text{as stated.} \end{aligned}$$

(4)

(b) $T = 5 + W$ with probability 1.

Thus

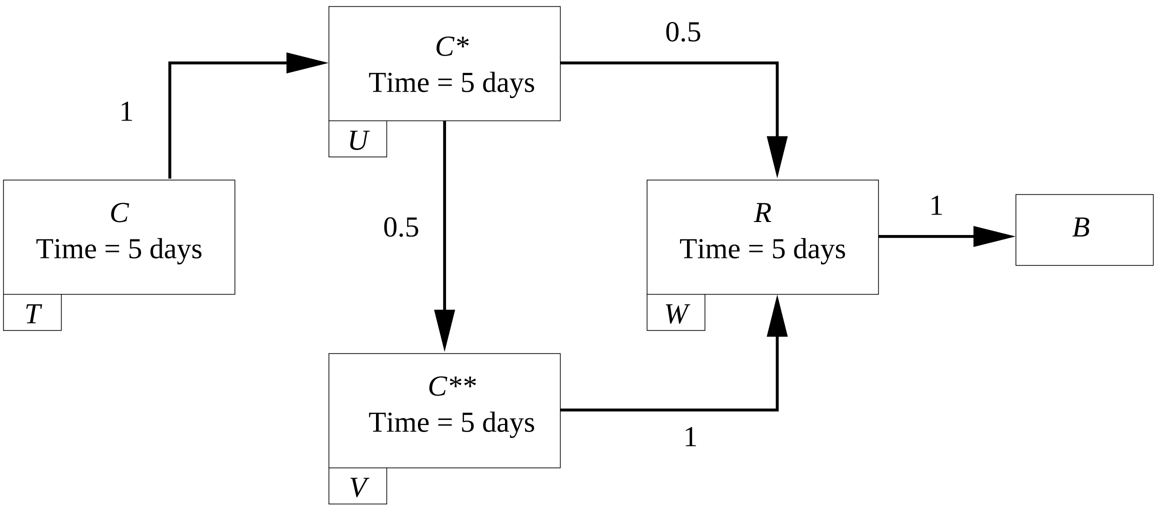
$$\begin{aligned} G(s) \;=\; \mathbb{E}(s^T) &= \mathbb{E}(s^{5+W}) \\ &= s^5\mathbb{E}(s^W) \\ &= s^5 H(s) \qquad \text{by definition of } H(s) \\ &= \frac{s^{10}}{2 - s^5}, \end{aligned}$$

using the answer to (a). (2)

(c) $G'(1) = \mathbb{E}(T)$, the expected number of days from entering state C to next entering state C. This is E_C from Question 1. (1)

3.(a) The arrows and probabilities are marked on the diagram.

Also marked on the diagram are times T, U, V, W, representing the *total* time taken to reach B, *starting* from the marked box.



(4)

(b) Starting from the states closest to B and using the notation on the diagram above:

$$\begin{aligned}
\mathbb{E}(W) &= 5 \text{ days.} \\
\mathbb{E}(V) &= 5 + \mathbb{E}(W) = 10 \text{ days.} \\
\mathbb{E}(U) &= 5 + \frac{1}{2}\mathbb{E}(V) + \frac{1}{2}\mathbb{E}(W) = 5 + \frac{10}{2} + \frac{5}{2} = 12.5 \text{ days.} \\
\mathbb{E}(T) &= 5 + \mathbb{E}(U) = 5 + 12.5 = 17.5 \text{ days.}
\end{aligned}$$

The required answer is therefore:

$$\mathbb{E}(T) = 17.5 \text{ days.}$$

(3)

4.(a) The probability function is:

y	0	5
$\mathbb{P}(Y=y)$	0.8	0.2

(1)

(b)

$$G(s) = 0.8 + 0.2s^5.$$

(1)

(c)

$$\begin{aligned} G_2(s) &= G\Big(G(s)\Big) \\ &= 0.8 + 0.2\Big(G(s)\Big)^5 \\ &= 0.8 + 0.2\Big(0.8 + 0.2s^5\Big)^5. \end{aligned}$$

(3)

(d)

$$\mathbb{P}(Z_2 = 0) = G_2(0) = 0.8 + 0.2\Big(0.8\Big)^5 = 0.866.$$

(2)

(e) We need $\mathbb{E}(Z_2) = G_2'(1)$. Now

$$G_2'(s) = 0.2 \times 5\Big(0.8 + 0.2s^5\Big)^4 \times 0.2 \times 5s^4,$$

so

$$G_2'(1) = 0.2 \times 5\Big(0.8 + 0.2\Big)^4 \times 1 = 1.$$

Thus

$$\mathbb{E}(Z_2) = G_2'(1) = 1.$$

(3)

Total: 30