1.(a) The expressions are:

$$\begin{aligned} E_S &= 1 + E_M \\ E_M &= 1 + \tfrac{1}{4}E_M + \tfrac{3}{4}E_D \\ E_D &= 1 + \tfrac{1}{2}E_D + \tfrac{1}{4}E_M + \tfrac{1}{4} \times 0 \end{aligned}$$

Basic simplification gives the following equations:

$$E_S = 1 + E_M \qquad (1)$$

$$3E_M = 4 + 3E_D \qquad (2)$$

$$E_D = 2 + \tfrac{1}{2}E_M \qquad (3)$$

Solving:

$$\begin{aligned} \text{(3) in (2):} \quad 3E_M &= 4 + 3\left(2 + \tfrac{1}{2}E_M\right) \\ \Rightarrow \quad \left(3 - \tfrac{3}{2}\right)E_M &= 4 + 3 \times 2 \\ E_M &= \frac{20}{3} = 6.67 \text{ courses.} \end{aligned}$$

Substituting in (3),

$$E_D = 2 + \frac{1}{2} \times \frac{20}{3} = \frac{16}{3} = 5.33 \text{ courses, as stated.}$$

Substituting in (1):

$$E_S = 1 + \frac{20}{3} = \frac{23}{3} = 7.67 \text{ courses in total.}$$

The final answers are:

$$E_S = 7.67, \qquad E_M = 6.67, \qquad E_D = 5.33.$$

(8)

(b) The Main and Dessert courses each take an average of 5 minutes. Only Main and Dessert courses occur after the first entry into state Main.

Thus the total expected time, starting from first entry into state Main, is 5 minutes times the total expected #courses, starting from first entry into state Main:

$$\mathbb{E}\left\{\text{time starting from first entry into state Main}\right\} = 5E_M = 5 \times \frac{20}{3} = \frac{100}{3} \text{ minutes.}$$

For the total expected time, we just need to add the 3 minutes expected time for one Soup course:

$$\mathbb{E}\left\{\text{total time}\right\} = 3 + \frac{100}{3} = \frac{109}{3} = 36.3 \text{ minutes.}$$

(3)

(c) Define the following expectations:

$$\begin{aligned}
D_S &= \mathbb{E}\{\#\text{desserts before stop, starting at entry into state Soup}\}\\
D_M &= \mathbb{E}\{\#\text{desserts before stop, starting at entry into state Main}\}\\
D_D &= \mathbb{E}\{\#\text{desserts before stop, starting at entry into state Dessert}\}\,.
\end{aligned}$$

Using first-step analysis, the equations are:

$$\begin{aligned}
D_S &= D_M\\
D_M &= \tfrac{1}{4}D_M + \tfrac{3}{4}D_D\\
D_D &= 1 + \tfrac{1}{2}D_D + \tfrac{1}{4}D_M
\end{aligned}$$

Basic simplification gives the following:

$$D_S = D_M \tag{4}$$

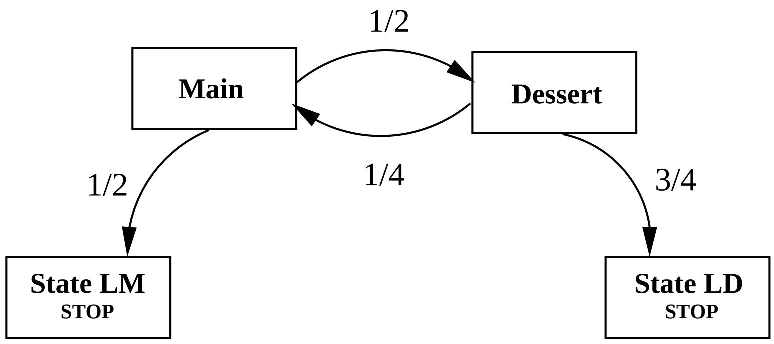
$$D_M = D_D \tag{5}$$

$$D_D = 1 + \tfrac{1}{2}D_D + \tfrac{1}{4}D_M \tag{6}$$

$$\begin{aligned}
\text{Substitute (5) into (6)} \quad \Rightarrow \quad D_D &= 1 + \tfrac{1}{2}D_D + \tfrac{1}{4}D_D\\
\Rightarrow \quad \tfrac{1}{4}D_D &= 1\\
\therefore \quad D_D &= 4 \text{ desserts.}
\end{aligned}$$

The expected number of desserts that the student eats is $D_S = 4$. **(4)**

2.(a) The redrawn diagram is below:



Define the following probabilities:

$$\begin{aligned}
p_M &= \mathbb{P}\{\text{end in state } LM\text{, starting from state Main}\}\\
p_D &= \mathbb{P}\{\text{end in state } LM\text{, starting from state Dessert}\}\,.
\end{aligned}$$

We wish to find p_M. Using first-step analysis, the equations are:

$$\begin{aligned}
p_M &= \tfrac{1}{2} + \tfrac{1}{2}p_D\\
p_D &= \tfrac{1}{4}p_M
\end{aligned}$$

Substituting $p_D = \frac{1}{4} p_M$ into the expression for p_M:

$$\begin{aligned} p_M &= \tfrac{1}{2} + \tfrac{1}{2} \times \tfrac{1}{4} p_M \\ \Rightarrow \quad \left(1 - \tfrac{1}{8}\right) p_M &= \tfrac{1}{2} \\ \Rightarrow \quad p_M &= \tfrac{4}{7}. \end{aligned}$$

The answer is:

$$p_M = \mathbb{P}(\text{last course eaten is a Main}) = \frac{4}{7}\,.$$

(6)

(b)

$$\mathbb{P}(\#\text{desserts} > 1) \;=\; 1 - \mathbb{P}(\#\text{desserts} = 0) - \mathbb{P}(\#\text{desserts} = 1)\,.$$

Using the diagram:

$$\mathbb{P}(\#\text{desserts} = 0) \;=\; \tfrac{1}{2}.$$

Similarly, to get exactly one dessert, we must enter state Dessert exactly once, following which we must either stop immediately, or return to state Main and then stop. So:

$$\begin{aligned} \mathbb{P}(\#\text{desserts} = 1) &= \tfrac{1}{2}\left(\tfrac{3}{4} + \tfrac{1}{4} \times \tfrac{1}{2}\right) \\ &= \tfrac{7}{16}. \end{aligned}$$

Thus:

$$\begin{aligned} \mathbb{P}(\#\text{desserts} > 1) &= 1 - \mathbb{P}(\#\text{desserts} = 0) - \mathbb{P}(\#\text{desserts} = 1) \\ &= 1 - \tfrac{1}{2} - \tfrac{7}{16} \\ &= \tfrac{1}{16}\,. \end{aligned}$$

The probability that the lecturer eats more than one dessert is $\frac{1}{16}$. **(3)**

Total: 24