1. The equations are:

$$\begin{aligned} p_M &= qp_U + rp_D \qquad \text{(a)} \\ p_U &= q + rp_M \qquad \text{(b)} \\ p_D &= qp_M \qquad \text{(c)} \end{aligned}$$

Substituting (b) and (c) in (a):

$$\begin{aligned} p_M &= q(q + rp_M) + rqp_M \\ p_M(1 - 2qr) &= q^2 \\ p_M &= \frac{q^2}{1 - 2qr} \qquad \text{as stated.} \end{aligned}$$

(4)

2.(a) Define the following expectations:

$$\begin{aligned} E_B &= \mathbb{E}\{\text{\#cups of tea before stop, starting at entry into state Bird}\} \\ E_C &= \mathbb{E}\{\text{\#cups of tea before stop, starting at entry into state Cup of Tea}\} \\ E_S &= \mathbb{E}\{\text{\#cups of tea before stop, starting at entry into state Serpent}\}. \end{aligned}$$

We want to find E_B. Using first-step analysis, the equations are:

$$\begin{aligned} E_B &= \tfrac{2}{6}E_C + \tfrac{3}{6}E_S \qquad \text{(a)} \\ E_C &= 1 + E_S \qquad \text{(b)} \\ E_S &= \tfrac{2}{3}E_B \qquad \text{(c)} \end{aligned}$$

Substituting (c) into (a):

$$\begin{aligned} E_B &= \tfrac{2}{6}E_C + \tfrac{3}{6} \times \tfrac{2}{3}E_B \\ \Rightarrow \quad 2E_B &= E_C \qquad \text{(d)} \end{aligned}$$

Substituting (d) and (c) into (b):

$$\begin{aligned} 2E_B &= 1 + \tfrac{2}{3}E_B \\ \Rightarrow \quad \tfrac{4}{3}E_B &= 1 \\ \Rightarrow \quad E_B &= \tfrac{3}{4} \quad (0.75). \end{aligned}$$

(6)

(b) Define the following probabilities:

$$\begin{aligned} p_B &= \mathbb{P}\{\text{no cups of tea before stop, starting at entry into state Bird}\} \\ p_S &= \mathbb{P}\{\text{no cups of tea before stop, starting at entry into state Serpent}\}. \end{aligned}$$

We wish to find p_B. The equations are:

$$\begin{aligned} p_B &= \tfrac{1}{6} + \tfrac{3}{6}p_S \quad \Rightarrow \quad 6p_B = 1 + 3p_S \qquad (a) \\ p_S &= \tfrac{2}{3}p_B + \tfrac{1}{3} \quad \Rightarrow \quad 3p_S = 2p_B + 1 \qquad (b) \end{aligned}$$

Substituting (b) in (a):

$$\begin{aligned} 6p_B &= 1 + 2p_B + 1 \\ \Rightarrow \quad p_B &= \tfrac{1}{2} \qquad (0.5). \end{aligned}$$

(4)

(c) By inspecting the diagram:

$$N = \begin{cases} 1 & \text{with probability } 1/6 \\ 1 + M & \text{with probability } 5/6 \end{cases} \qquad M = \begin{cases} 1 & \text{with probability } 1/3 \\ 1 + N' & \text{with probability } 2/3 \end{cases}$$

where $N' \sim N$. Thus

$$\mathbb{E}(s^N) = \begin{cases} s & \text{w.p. } 1/6 \\ s\mathbb{E}(s^M) & \text{w.p. } 5/6 \end{cases} \qquad \mathbb{E}(s^M) = \begin{cases} s & \text{w.p. } 1/3 \\ s\mathbb{E}(s^N) & \text{w.p. } 2/3 \end{cases}$$

So

$$\begin{aligned} G(s) = \mathbb{E}(s^N) &= \tfrac{1}{6}s + \tfrac{5}{6}sH(s) \quad \Rightarrow \quad 6G(s) = s + 5sH(s) \qquad \text{(a)} \\ H(s) = \mathbb{E}(s^M) &= \tfrac{1}{3}s + \tfrac{2}{3}sG(s) \qquad \text{(b)} \end{aligned}$$

Substituting (b) in (a):

$$\begin{aligned} 6G(s) &= s + 5s\left\{\tfrac{1}{3}s + \tfrac{2}{3}sG(s)\right\} \\ \Rightarrow \quad 18G(s) &= 3s + 5s^2 + 10s^2G(s) \\ \Rightarrow \quad G(s)(18 - 10s^2) &= 3s + 5s^2 \\ \Rightarrow \quad G(s) &= \frac{3s + 5s^2}{18 - 10s^2} \qquad \text{as stated.} \end{aligned}$$

Substituting this in (b):

$$\begin{aligned} H(s) &= \tfrac{1}{3}s + \tfrac{2}{3}s \times \frac{3s + 5s^2}{18 - 10s^2} \\ &= \frac{s}{3} \times \frac{(18 - 10s^2) + (6s + 10s^2)}{18 - 10s^2} \\ &= \frac{6s + 2s^2}{18 - 10s^2} \end{aligned}$$

(6)

Total: (20)