(a) Define the following expectations:

$$\begin{aligned} m_K &= \mathbb{E}\{\text{time to finish, starting at entry into Kitchen}\} \\ m_B &= \mathbb{E}\{\text{time to finish, starting at entry into Bedroom}\} \\ m_L &= \mathbb{E}\{\text{time to finish, starting at entry into Lounge}\} \end{aligned}$$

Using first-step analysis, the equations are:

$$\begin{aligned} m_K &= 2 + \tfrac{2}{5}m_B + \tfrac{2}{5}m_L && \text{(a)} \\ m_B &= 15 + m_K && \text{(b)} \\ m_L &= 4 + m_K && \text{(c)} \end{aligned}$$

Substituting (b) and (c) into (a):

$$\begin{aligned} m_K &= 2 + \tfrac{2}{5}(15 + m_K) + \tfrac{2}{5}(4 + m_K) \\ \Rightarrow \quad 5m_K &= 10 + 30 + 2m_K + 8 + 2m_K \\ \Rightarrow \quad m_K &= 48 \text{ minutes.} \end{aligned}$$

(4)

(b) Define the following expectations:

$$\begin{aligned} n_K &= \mathbb{E}\{\text{\# visits to bedroom, starting at entry into Kitchen}\} \\ n_B &= \mathbb{E}\{\text{\# visits to bedroom, starting at entry into Bedroom}\} \\ n_L &= \mathbb{E}\{\text{\# visits to bedroom, starting at entry into Lounge}\} \end{aligned}$$

We wish to find n_K. The equations are:

$$\begin{aligned} n_K &= \tfrac{2}{5}n_B + \tfrac{2}{5}n_L && \text{(a)} \\ n_B &= 1 + n_K && \text{(b)} \\ n_L &= n_K && \text{(c)} \end{aligned}$$

Substituting (b) and (c) into (a):

$$\begin{aligned} n_K &= \tfrac{2}{5} + \tfrac{2}{5}n_K + \tfrac{2}{5}n_K \\ \Rightarrow \quad n_K &= 2 \qquad \text{(Expected number of visits to bedroom.)} \end{aligned}$$

(4)

(c) $$\mathbb{E}(\text{\# escapes}) = \mathbb{E}(\text{\# visits to bedroom}) + \mathbb{E}(\text{\# visits to lounge}).$$

The diagram is symmetrical in probabilities, so

$$\mathbb{E}(\text{\# visits to lounge}) = \mathbb{E}(\text{\# visits to bedroom}) = 2 \text{ from part (b).}$$

Thus

$$\mathbb{E}(\text{\# escapes}) = 2 + 2 = 4.$$

(2)

(d) N is the number of visits to the bedroom, starting at the kitchen. By inspecting the diagram:

$$N = \begin{cases} 1 + N' & \text{with probability } 2/5 \\ 0 & \text{with probability } 1/5 \\ N'' & \text{with probability } 2/5 \end{cases}$$

where $N' \sim N'' \sim N$. Thus

$$\mathbb{E}(s^N) = \begin{cases} s\mathbb{E}(s^N) & \text{with probability } 2/5 \\ 1 & \text{with probability } 1/5 \\ \mathbb{E}(s^N) & \text{with probability } 2/5 \end{cases}$$

So

$$\begin{aligned} G(s) = \mathbb{E}(s^N) &= \tfrac{2}{5}sG(s) + \tfrac{1}{5} + \tfrac{2}{5}G(s) \\ \Rightarrow \quad 3G(s) - 2sG(s) &= 1 \\ \Rightarrow \quad G(s) &= \frac{1}{3-2s} \quad \text{as stated.} \end{aligned}$$

(3)

(e)

$$G(s) = \frac{1}{3-2s} = \frac{1/3}{3/3 - (2/3)s} = \frac{1/3}{1-(2/3)s}\,.$$

This is the PGF of the Geometric($p = \frac{1}{3}$) distribution, so $N \sim \text{Geometric}\left(p = \frac{1}{3}\right)$. Thus the probability that Mrs Figg and the cat never visit the bedroom is

$$\mathbb{P}(N = 0) = \left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)^0 = \tfrac{1}{3}\,.$$

(3)

(f) By inspection, the state at 10 minutes is partitioned by the following possibilities:

Steps	State at 10 minutes	Probability
$K \to B$	Bedroom	$\frac{2}{5}$
$K \to$ Success	Finished	$\frac{1}{5}$
$K \to L \to K \to B$	Bedroom	$\frac{2}{5} \times \frac{2}{5} = \frac{4}{25}$
$K \to L \to K \to$ Success	Finished	$\frac{2}{5} \times \frac{1}{5} = \frac{2}{25}$
$K \to L \to K \to L$	Lounge	$\frac{2}{5} \times \frac{2}{5} = \frac{4}{25}$

Thus

$$\begin{aligned} \mathbb{P}(\text{in bedroom after 10 minutes}) &= \tfrac{2}{5} + \tfrac{4}{25} = \tfrac{14}{25} \\ \mathbb{P}(\text{in lounge after 10 minutes}) &= \tfrac{4}{25} \\ \mathbb{P}(\text{finished after 10 minutes}) &= \tfrac{1}{5} + \tfrac{2}{25} = \tfrac{7}{25}. \end{aligned}$$

(4)

Total: (20)