1.(a) Define the following probabilities:

$$\begin{aligned} p_A &= \mathbb{P}(\text{last song played is } A \mid \text{start at the beginning of Song } A) \\ p_B &= \mathbb{P}(\text{last song played is } A \mid \text{start at the beginning of Song } B) \\ p_C &= \mathbb{P}(\text{last song played is } A \mid \text{start at the beginning of Song } C) \end{aligned}$$

Using first-step analysis, the equations are:

$$\begin{aligned} p_A &= \tfrac{3}{4}p_B + \tfrac{1}{4} \times 1 && \text{(a)} \\ p_B &= \tfrac{1}{4}p_A + \tfrac{3}{4}p_C && \text{(b)} \\ p_C &= \tfrac{3}{4} \times 0 + \tfrac{1}{4}p_B && \text{(c)} \end{aligned}$$

Substituting (a) and (c) into (b):

$$\begin{aligned} p_B &= \tfrac{1}{4}\left(\tfrac{3}{4}p_B + \tfrac{1}{4}\right) + \tfrac{3}{4}\left(\tfrac{1}{4}p_B\right) \\ \Rightarrow \quad 16p_B &= 6p_B + 1 \\ \Rightarrow \quad p_B &= \tfrac{1}{10} \\ \text{Substitute in (a)} \quad \Rightarrow \quad p_A &= \tfrac{13}{40} = 0.325. \end{aligned}$$

The probability that A is the last song played, starting from the beginning, is $p_A = 0.325$.

(4)

(b) Define the following expectations:

$$\begin{aligned} e_A &= \mathbb{E}\{\#\text{ times song } B \text{ played} \mid \text{start at the beginning of Song } A\} \\ e_B &= \mathbb{E}\{\#\text{ times song } B \text{ played} \mid \text{start at the beginning of Song } B\} \\ e_C &= \mathbb{E}\{\#\text{ times song } B \text{ played} \mid \text{start at the beginning of Song } C\} \end{aligned}$$

Note that $\mathbb{E}(N) = e_A$. Using first-step analysis, the equations are:

$$\begin{aligned} e_A &= pe_B + (1-p) \times 0 && \text{(a)} \\ e_B &= 1 + pe_C + (1-p)e_A && \text{(b)} \\ e_C &= (1-p)e_B + p \times 0 && \text{(c)} \end{aligned}$$

Substituting (a) and (c) into (b):

$$\begin{aligned} e_B &= 1 + p(1-p)e_B + (1-p)pe_B \\ \Rightarrow \quad e_B\{1 - 2p(1-p)\} &= 1 \\ \Rightarrow \quad e_B &= \frac{1}{1 - 2p + 2p^2} \end{aligned}$$

$$\text{Substitute in (a)} \quad \Rightarrow \quad e_A = \mathbb{E}(N) = \frac{p}{1-2p+2p^2}, \text{ as stated.}$$

(4)

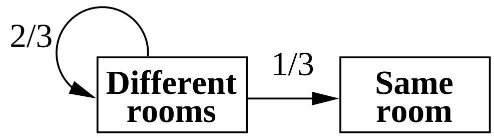
(c) For $\mathbb{E}(N) = 1$, we require:

$$\begin{aligned} 1 &= \frac{p}{1-2p+2p^2} \\ \Rightarrow \quad 2p^2 - 3p + 1 &= 0 \\ \Rightarrow \quad (2p-1)(p-1) &= 0. \end{aligned}$$

The two values of p that ensure $\mathbb{E}(N) = 1$ are $p = \frac{1}{2}$ and $p = 1$.

When $p = 1$, the process goes $A \to B \to C \to$ Stop with certainty, so $N = 1$ always and $\text{Var}(N) = 0$. When $p = \frac{1}{2}$, N is not constant. Thus $p = \frac{1}{2}$ will give the higher variance for N. **(4)**

2.(a) All states are symmetric, so we can imagine that the robber is in Room 1 and the policeman is in Room 2. At the next step, the robber will be in Room 2 with probability 1/3 and either Room 3 or 4 with probability 2/3 combined. The diagram is:



Define $m_1 = \mathbb{E}\{\text{number of steps to finish in Strategy 1} \mid \text{start in state Different}\}$, then

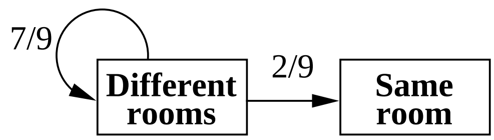
$$m_1 = 1 + \tfrac{2}{3}m_1 \quad \Rightarrow \quad m_1 = 3 \text{ steps.}$$

(3)

(b) Again, imagine that the robber is in Room 1 and the policeman is in Room 2. At the next step, we have the following possibilities, each with probability 1/9:

Robber	Policeman		
2	1	3	4
3	1	3	4
4	1	3	4

The robber and policeman are in the same room with probability 2/9: they could be both in Room 3 or both in Room 4. Thus the diagram is:



Define $m_2 = \mathbb{E}\{\text{number of steps to finish in Strategy 2} \mid \text{start in state Different}\}$, then

$$m_2 = 1 + \tfrac{7}{9}m_2 \quad \Rightarrow \quad m_2 = 9/2 = 4.5 \text{ steps.}$$

(4)

(c) We have $m_1 = 3$ and $m_2 = 4.5$. The policeman should use Strategy 1, because $m_1 < m_2$. **(1)**

Total: (20)