(a) Define the following expectations:

$$\begin{aligned} m_L &= \mathbb{E}\,(\text{number of balls before the end} \,|\, \text{start just before entering box L}) \\ m_R &= \mathbb{E}\,(\text{number of balls before the end} \,|\, \text{start just before entering box R}) \end{aligned}$$

Using first-step analysis, the equations are:

$$\begin{aligned} m_L &= 1 + \tfrac{1}{6} m_L + \tfrac{3}{6} m_R \qquad &\text{(a)} \\ m_R &= 1 + \tfrac{2}{4} m_L + \tfrac{1}{4} m_R \qquad &\text{(b)} \end{aligned}$$

Rearranging (b):

$$\begin{aligned} \tfrac{3}{4} m_R &= 1 + \tfrac{2}{4} m_L \\ \Rightarrow \quad m_R &= \tfrac{4}{3} + \tfrac{2}{3} m_L. \qquad \text{(c)} \end{aligned}$$

Substituting (c) in (a):

$$\begin{aligned} \tfrac{5}{6} m_L &= 1 + \tfrac{2}{3} + \tfrac{1}{3} m_L \\ \tfrac{3}{6} m_L &= \tfrac{10}{6} \\ m_L &= \tfrac{10}{3} \end{aligned}$$

So also, from (c):

$$m_R = \tfrac{32}{9}$$

(5)

(b) From the diagram,

$$\mathbb{E}(N) = \tfrac{1}{2} m_L + \tfrac{1}{2} m_R = \tfrac{1}{2}\left\{\tfrac{10}{3} + \tfrac{32}{9}\right\} = \tfrac{31}{9}.$$

(1)

(c)

$$\mathbb{P}(\text{L, R, R, Stop}) = \tfrac{1}{2} \times \tfrac{3}{6} \times \tfrac{1}{4} \times \tfrac{1}{4} = \tfrac{1}{64} \quad (0.0156).$$

(2)

(d) Define the following probabilities:

$$\begin{aligned} p_S &= \mathbb{P}\,(\text{get no Lefts before the end} \,|\, \text{ start at state } \textbf{Start}) \\ p_R &= \mathbb{P}\,(\text{get no Lefts before the end} \,|\, \text{ start at state R}) \end{aligned}$$

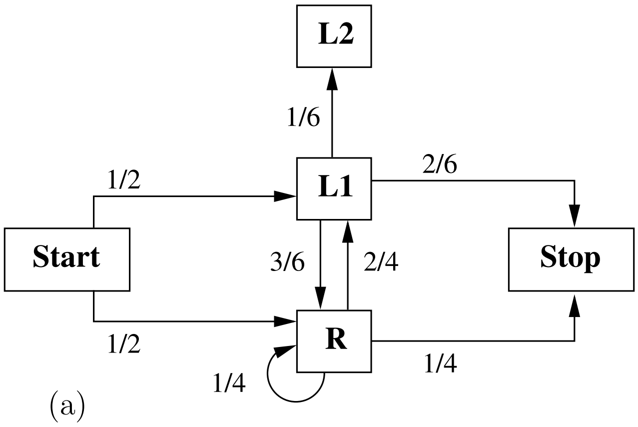
First-step analysis:

$$\begin{aligned} p_S &= \tfrac{1}{2} p_R + \tfrac{1}{2} \times 0 \\ p_R &= \tfrac{1}{4} p_R + \tfrac{1}{4} \times 1 + \tfrac{2}{4} \times 0 \\ \Rightarrow \quad \tfrac{3}{4} p_R &= \tfrac{1}{4} \\ p_R &= \tfrac{1}{3} \\ \therefore \quad p_S &= \tfrac{1}{6}. \end{aligned}$$

The required probability is $p_s = \frac{1}{6}$. **(3)**

(e) We need to define a new state, L_2, that describes getting the second Left in a row. Let L_1 be a state describing the first Left in any sequence of Lefts. We can terminate the diagram at L_2 because our interest is only in whether we hit state L_2 before we hit state **Stop**: we don't care what happens after L_2.

The new diagram is here.



Define the following probabilities:

$h_S = \mathbb{P}(\text{hit state } L_2 \mid \text{ start at } \textbf{Start})$
$h_L = \mathbb{P}(\text{hit state } L_2 \mid \text{ start at } L_1)$
$h_R = \mathbb{P}(\text{hit state } L_2 \mid \text{ start at } R)$

First-step analysis:

$$
\begin{aligned}
h_S &= \tfrac{1}{2}h_L + \tfrac{1}{2}h_R && \text{(a)} \\
h_L &= \tfrac{1}{6} + \tfrac{3}{6}h_R && \text{(b)} \\
h_R &= \tfrac{2}{4}h_L + \tfrac{1}{4}h_R && \text{(c)}
\end{aligned}
$$

$$
\begin{aligned}
\text{(c)} \quad &\Rightarrow & h_R &= \tfrac{2}{3}h_L \\
\text{Subst in (b)} \quad &\Rightarrow & h_L &= \tfrac{1}{6} + \tfrac{1}{3}h_L \\
&\Rightarrow & h_L &= \tfrac{1}{4} \\
&\Rightarrow & h_R &= \tfrac{1}{6}
\end{aligned}
$$

Thus the probability of ever getting two lefts in a row is $h_S = \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{6} = \frac{5}{24}$ (0.208).

(6)

(f) **False:** The new diagram can **not** give the same distribution of N as the first diagram, for any p and q. The reason is that Left and Right balls do need to be distinguished. The probability of stopping after a Left ball (2/6) is different from the probability of stopping after a Right ball (1/4), so the two states can not be treated as having an identical stopping probability q.

To be sure that there is no p and q that can give the same distribution of N, we can show that we would need $q = \frac{1}{\mathbb{E}(N)} = \frac{9}{31}$, $p = \frac{22}{31}$ to match the value of $\mathbb{E}(N)$ given in (b). However, with these values of p and q, we would get $\mathbb{P}(N = 1) = \frac{9}{31}$ with the new diagram, but $\mathbb{P}(N = 1) = \frac{1}{2}\left(\frac{2}{6} + \frac{1}{4}\right) = \frac{7}{24}$ for the old diagram. These are not the same, so no values of p and q can give the required distribution of N. **(3)**

Total: (20)