(a) Define the following expectations:

$$\begin{aligned} m_B &= \mathbb{E}\,(\text{number of times Baby passes the ball} \,|\, \text{start at state B}) \\ m_M &= \mathbb{E}\,(\text{number of times Baby passes the ball} \,|\, \text{start at state M}) \\ m_D &= \mathbb{E}\,(\text{number of times Baby passes the ball} \,|\, \text{start at state D}) \end{aligned}$$

First-step analysis equations: (note we add 1 only when Baby passes the ball)

$$\begin{aligned} m_B &= 1 + \tfrac{1}{2} m_M + \tfrac{1}{2} m_D \qquad &(a) \\ m_M &= \tfrac{2}{5} m_B + \tfrac{1}{5} m_D \qquad &(b) \\ m_D &= \tfrac{3}{5} m_B + \tfrac{1}{5} m_M \qquad &(c) \end{aligned}$$

Substitute (c) in (b):

$$\begin{aligned} m_M &= \tfrac{2}{5} m_B + \tfrac{1}{5}\left(\tfrac{3}{5} m_B + \tfrac{1}{5} m_M\right) \\ \Rightarrow \quad m_M &= \tfrac{13}{24} m_B. \qquad (d) \\ \text{In (c):} \quad m_D &= \left(\tfrac{3}{5} + \tfrac{1}{5} \times \tfrac{13}{24}\right) m_B \\ \Rightarrow \quad m_D &= \tfrac{17}{24} m_B. \qquad (e) \end{aligned}$$

Substitute (d) and (e) in (a):

$$\begin{aligned} m_B &= 1 + \tfrac{1}{2}\left(\tfrac{13}{24} + \tfrac{17}{24}\right) m_B \\ \Rightarrow \quad m_B &= \tfrac{8}{3} \end{aligned}$$

The final answer is

$$m_B = \mathbb{E}\,(\text{number of times Baby passes the ball} \,|\, \text{start at state B}) = \tfrac{8}{3} = 2.67 \text{ times.}$$

(6)

(b)

$$\mathbb{P}(\text{B, D, M, Stop}) = 1 \times \tfrac{1}{2} \times \tfrac{1}{5} \times \tfrac{2}{5} = \tfrac{1}{25} = 0.04.$$

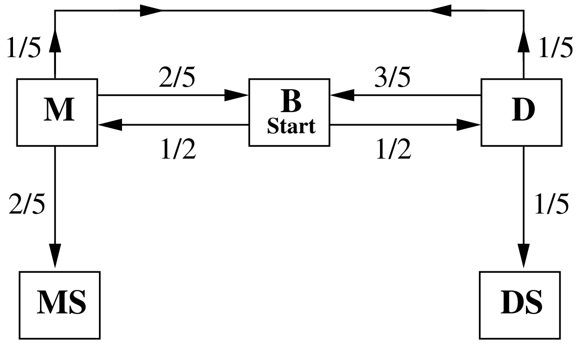
(2)

(c) There are only two trajectories allowing only one pass: they are B, M, Stop and B, D, Stop. Thus

$$\begin{aligned}\mathbb{P}(\text{only one pass}) &= \mathbb{P}(\text{B, M, Stop}) + \mathbb{P}(\text{B, D, Stop}) \\ &= \tfrac{1}{2} \times \tfrac{2}{5} + \tfrac{1}{2} \times \tfrac{1}{5} \\ &= \tfrac{3}{10} \quad (0.3).\end{aligned}$$

(2)

(d) (i) The new diagram distinguishes between the case where Mum stops the game (state MS), and the case where Dad stops the game (state DS).



(ii) Define the following probabilities:

$$\begin{aligned}p_M &= \mathbb{P}\,(\text{ends in state MS} \mid \text{ start at state M}) \\ p_B &= \mathbb{P}\,(\text{ends in state MS} \mid \text{ start at state B}) \\ p_D &= \mathbb{P}\,(\text{ends in state MS} \mid \text{ start at state D})\end{aligned}$$

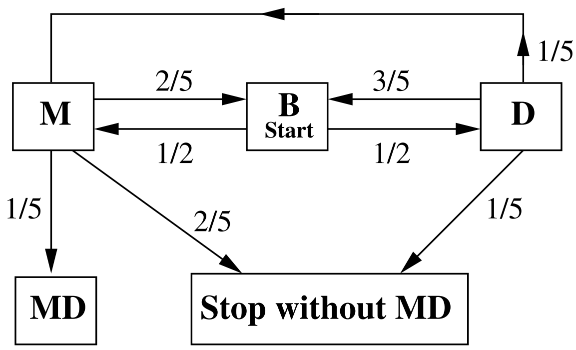
(iii) First-step analysis equations:

$$\begin{aligned}p_M &= \tfrac{2}{5} + \tfrac{2}{5}p_B + \tfrac{1}{5}p_D \\ p_B &= \tfrac{1}{2}p_M + \tfrac{1}{2}p_D \\ p_D &= \tfrac{3}{5}p_B + \tfrac{1}{5}p_M\end{aligned}$$

Complete the solution by solving for p_B. **(5)**

(e) Let state MD denote a pass from Mum to Dad. We can terminate the process here because it is the state of interest.

Diagram:



Define the notation:

$$
\begin{aligned}
p_M &= \mathbb{P}(\text{ends in state MD} \mid \text{ start at state M}) \\
p_B &= \mathbb{P}(\text{ends in state MD} \mid \text{ start at state B}) \\
p_D &= \mathbb{P}(\text{ends in state MD} \mid \text{ start at state D})
\end{aligned}
$$

First-step analysis equations:

$$
\begin{aligned}
p_M &= \tfrac{1}{5} + \tfrac{2}{5}p_B && \text{(a)} \\
p_B &= \tfrac{1}{2}p_M + \tfrac{1}{2}p_D && \text{(b)} \\
p_D &= \tfrac{3}{5}p_B + \tfrac{1}{5}p_M && \text{(c)}
\end{aligned}
$$

Substitute (c) in (b):

$$
\begin{aligned}
p_B &= \tfrac{1}{2}p_M + \tfrac{1}{2}\left(\tfrac{3}{5}p_B + \tfrac{1}{5}p_M\right) \\
\Rightarrow \quad p_M &= \tfrac{7}{6}\,p_B && \text{(d)}
\end{aligned}
$$

Substitute (d) in (a):

$$
\begin{aligned}
\tfrac{7}{6}\,p_B &= \tfrac{1}{5} + \tfrac{2}{5}p_B \\
\Rightarrow \quad \tfrac{23}{30}\,p_B &= \tfrac{1}{5} \\
p_B &= \tfrac{6}{23}
\end{aligned}
$$

The final answer is:

$$
p_B = \mathbb{P}(\text{ever have a pass from Mum to Dad} \mid \text{ start at state B}) = \frac{6}{23} \qquad (0.261).
$$

(5)

Total: (20)