(a) Define the following probabilities:

$$\begin{aligned} p_G &= \mathbb{P}\,(\text{Hobbit wins} \,|\, \text{start at state } G) \\ p_D &= \mathbb{P}\,(\text{Hobbit wins} \,|\, \text{start at state } D) \\ p_T &= \mathbb{P}\,(\text{Hobbit wins} \,|\, \text{start at state } T) \end{aligned}$$

First-step analysis equations:

$$\begin{aligned} p_G &= \tfrac{2}{5}p_T + \tfrac{2}{5}p_D && \text{(a)} \\ p_D &= \tfrac{2}{5}p_T && \text{(b)} \\ p_T &= \tfrac{1}{2}p_G + \tfrac{1}{2} && \text{(c)} \end{aligned}$$

Substitute (c) in (b):

$$p_D = \tfrac{1}{5}p_G + \tfrac{1}{5} \qquad \text{(d)}$$

Substitute (b) and (d) in (a):

$$\begin{aligned} p_G &= \tfrac{7}{5}p_D \\ &= \tfrac{7}{5}\left\{\tfrac{1}{5}p_G + \tfrac{1}{5}\right\} \\ \Rightarrow \quad \tfrac{18}{25}p_G &= \tfrac{7}{25} \\ \Rightarrow \quad p_G &= \tfrac{7}{18}\,. \end{aligned}$$

The final answer is:

$$p_G = \mathbb{P}\,(\text{Hobbit wins} \,|\, \text{start at state } G) = \tfrac{7}{18} = 0.389.$$

We also obtain $p_D = \frac{5}{18}$ and $p_T = \frac{25}{36}$. **(8)**

(b) Define the following probabilities:

$$\begin{aligned} h_G &= \mathbb{P}\,(\text{Hobbit ever meets the Dragon} \,|\, \text{start at state } G) \\ h_T &= \mathbb{P}\,(\text{Hobbit ever meets the Dragon} \,|\, \text{start at state } T) \end{aligned}$$

First-step analysis equations:

$$\begin{aligned} h_G &= \tfrac{2}{5} + \tfrac{2}{5}h_T && \text{(a)} \\ h_T &= \tfrac{1}{2}h_G && \text{(b)} \end{aligned}$$

Solving:

$$h_G = \tfrac{2}{5} + \tfrac{1}{5}h_G \quad \Rightarrow \quad h_G = \tfrac{1}{2}.$$

The probability that the Hobbit ever meets the Dragon, starting from the beginning, is $h_G = \frac{1}{2}$. **(6)**

(c) Using Bayes' Theorem:

$$\mathbb{P}(D \mid W) \;=\; \frac{\mathbb{P}(W \mid D)\mathbb{P}(D)}{\mathbb{P}(W)}$$

Now:

$\mathbb{P}(W \mid D) \;=\; p_D \;=\; \frac{5}{18}$ from (a): prob of winning having met the Dragon;

$\mathbb{P}(D) \;=\; h_G \;=\; \frac{1}{2}$ from (b): prob of meeting the Dragon, starting from beginning;

$\mathbb{P}(W) \;=\; p_G \;=\; \frac{7}{18}$ from (a): prob of winning, starting from the beginning.

So

$$\mathbb{P}(D \mid W) = \frac{\frac{5}{18} \times \frac{1}{2}}{\frac{7}{18}} = \frac{5}{14}\,.$$

The probability that the Hobbit ever meets the Dragon, given that he wins eventually, is $\frac{5}{14}$. **(6)**

(d) Define the following expectations:

$$\begin{aligned} m_G &= \mathbb{E}\,(\#\text{times Hobbit visits state Treasure} \mid \text{start at state } G) \\ m_D &= \mathbb{E}\,(\#\text{times Hobbit visits state Treasure} \mid \text{start at state } D) \\ m_T &= \mathbb{E}\,(\#\text{times Hobbit visits state Treasure} \mid \text{start at state } T) \end{aligned}$$

We seek $\mathbb{E}(N) = m_G$. We could count the visit to Treasure either on the arrows going into state T, *or* on the arrows going out of state T, but not both. It is easiest to use the arrows going out, as below, but either method is correct.

First-step analysis equations:

$$\begin{aligned} m_G &= \tfrac{2}{5}m_T + \tfrac{2}{5}m_D \quad \text{(a)} \\ m_D &= \tfrac{2}{5}m_T \quad \text{(b)} \\ m_T &= 1 + \tfrac{1}{2}m_G \quad \text{(c)} \end{aligned}$$

Substitute (b) in (a):

$$m_G \;=\; \left(\tfrac{2}{5} + \tfrac{4}{25}\right) m_T \;=\; \tfrac{14}{25}m_T \qquad \text{(d)}$$

Substitute (c) in (d):

$$\begin{aligned} m_G &= \tfrac{14}{25} + \tfrac{7}{25}m_G \\ \Rightarrow \quad \tfrac{18}{25}m_G &= \tfrac{14}{25} \\ \Rightarrow \quad m_G &= \tfrac{7}{9} \quad (0.778). \end{aligned}$$

The expected number of times that the Hobbit visits the state Treasure, starting from the beginning, is

$$\mathbb{E}(N) = m_G = \tfrac{7}{9}.$$

(6)

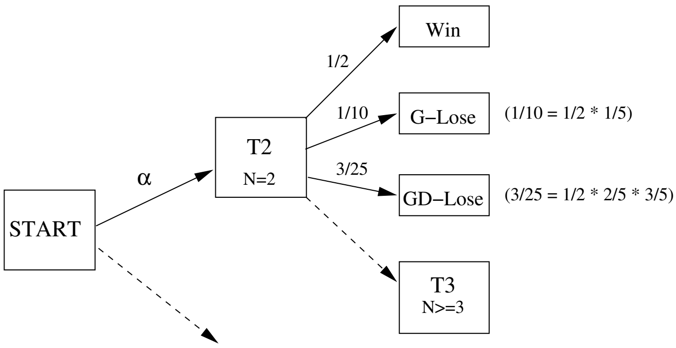
(e) For sample points that are members of the event $\{N = 2\}$, any path from start to end with **exactly two visits to state T** is acceptable. Examples:

$$\text{Start} \to G \to T \to G \to T \to \text{Win}$$

$$\text{Start} \to G \to D \to T \to G \to T \to G \to \text{Lose}$$

Therefore, conditioning on the event $\{N = 2\}$ means we are restricting attention to all paths with exactly two visits to state T.

For $\mathbb{P}(W \mid N = 2)$, consider a new diagram, starting at the beginning (Ω), and marking 'state $T2$' when we enter state T for the second time. There are only four possibilities on leaving state $T2$ before we either finish or enter state T for the third time (in which case $N = 3$ so we are no longer in the situation $N = 2$). Define probability α to be the probability of reaching state $T2$ from the beginning. (It doesn't matter what α is.) Multiplying out the probabilities of all paths from $T2$ to the end, keeping $N = 2$ exactly, gives the following diagram:



So

$$\mathbb{P}(W \mid N = 2) = \frac{\mathbb{P}(W \cap (N = 2))}{\mathbb{P}(N = 2)} = \frac{\alpha \times \frac{1}{2}}{\alpha \times (\frac{1}{2} + \frac{1}{10} + \frac{3}{25})} = \frac{25}{36}.$$

Therefore

$$\mathbb{P}(W \mid N = 2) = \tfrac{25}{36}.$$

Clearly, the same argument would apply for any integer $n \geq 1$:

$$\mathbb{P}(W \mid N = n) = \tfrac{25}{36} \quad \text{for any } n = 1, 2, 3, \ldots$$

(6)

(f) Definition of $\mathbb{E}(N \mid W)$ as a sum (remembering N is a *random variable* and W is an *event*):

$$\mathbb{E}(N \mid W) = \sum_{n=0}^{\infty} n\, \mathbb{P}(N = n \mid W).$$

Using Bayes' Theorem within the sum:

$$\begin{aligned} \mathbb{E}(N \mid W) &= \sum_{n=0}^{\infty} n\, \mathbb{P}(N = n \mid W) \\ &= \sum_{n=0}^{\infty} n \left(\frac{\mathbb{P}(W \mid N = n)\mathbb{P}(N = n)}{\mathbb{P}(W)} \right). \qquad (\star) \end{aligned}$$

The comment at the end of (e) remarks that

$$\mathbb{P}(W \mid N = n) = \tfrac{25}{36} \quad \text{for } n = 1, 2, 3, \ldots$$

It is also clear that $\mathbb{P}(W \mid N = 0) = 0$, because we have to go through state T in order to win. However, this term does not contribute anything to the sum because it is multiplied by $n = 0$.

Also, $\mathbb{P}(W) = p_G = \frac{7}{18}$ from part (a).

Continuing from ($\star$) and ignoring the zero-term in $n = 0$:

$$\begin{aligned}
\mathbb{E}(N \mid W) &= \sum_{n=1}^{\infty} n \left(\frac{\frac{25}{36}\mathbb{P}(N = n)}{\frac{7}{18}} \right) \\
&= \tfrac{25}{14} \sum_{n=1}^{\infty} n\, \mathbb{P}(N = n) \\
&= \tfrac{25}{14} \times \mathbb{E}(N) \qquad \text{noting that the term in } n = 0 \text{ does not contribute to } \mathbb{E}(N) \\
&= \tfrac{25}{14} \times \tfrac{7}{9} \qquad \text{from part (d)} \\
&= \tfrac{25}{18} \quad (1.389).
\end{aligned}$$

Therefore, the expected number of visits to Treasure, given that the Hobbit wins overall, is

$$\mathbb{E}(N \mid W) = \tfrac{25}{18} = 1.389.$$

Not surprisingly, this is greater than the unconditional expected number of visits to Treasure, which is $\mathbb{E}(N) = \frac{7}{9} = 0.778$ from part (d). **(8)**

Alternative method for part (f):

An alternative method uses the following two ideas:

1. $$\mathbb{P}(W \mid N \geq n) = p_T = \tfrac{25}{36} \quad \text{for any } n \geq 1,$$
   where p_T was calculated in part (a), and is the probability of winning eventually starting from state T. It is fairly obvious why this is true when you consider that all routes through the sample space that satisfy $N \geq n$ start when they enter state T for the n'th time, so the probability of winning from there is the same as the probability of winning from any entry to state T, which is $p_T = \frac{25}{36}$.
2. Using the argument featured in Bonus Question 4(b) (Exam 2009 Q7b):
   $$\mathbb{E}(N \mid W) = \sum_{n=1}^{\infty} n\mathbb{P}(N = n \mid W) = \sum_{n=1}^{\infty} \mathbb{P}(N \geq n \mid W).$$

Applying Bayes Theorem inside the sum in (2) and using (1) gives:

$$\mathbb{E}(N \mid W) = \sum_{n=1}^{\infty} \frac{\mathbb{P}(W \mid N \geq n)\mathbb{P}(N \geq n)}{\mathbb{P}(W)} = \frac{p_T}{p_G} \sum_{n=1}^{\infty} \mathbb{P}(N \geq n) = \frac{p_T \mathbb{E}(N)}{p_G} = \frac{25}{18} \text{ as before.}$$

Total: (40)