1.(a) Define the following probabilities:

$$\begin{aligned} p_S &= \mathbb{P}\,(\text{visitor attends the Lecture} \,|\, \text{start at state } S) \\ p_P &= \mathbb{P}\,(\text{visitor attends the Lecture} \,|\, \text{start at state } P) \\ p_R &= \mathbb{P}\,(\text{visitor attends the Lecture} \,|\, \text{start at state } R) \end{aligned}$$

First-step analysis equations:

$$\begin{aligned} p_S &= 0.3p_P + 0.7p_R && \text{(a)} \\ p_P &= 0.4p_P + 0.3p_R + 0.2 && \text{(b)} \\ p_R &= 0.5p_R + 0.2p_P + 0.1 && \text{(c)} \end{aligned}$$

Rearrange (b) and (c) and multiply by 10:

$$\begin{aligned} 6p_P &= 3p_R + 2 && (\text{b}^\star) \\ 5p_R &= 2p_P + 1 && (\text{c}^\star) \end{aligned}$$

Substitute $3 \times (\text{c}^\star)$ in $(\text{b}^\star)$:

$$\begin{aligned} 15p_R - 3 &= 3p_R + 2 \\ \Rightarrow \quad p_R &= \tfrac{5}{12} = \tfrac{10}{24} = 0.417. \end{aligned}$$

Substitute $p_R = \frac{10}{24}$ in $(\text{b}^\star)$:

$$\begin{aligned} 6p_P &= 3 \times \tfrac{10}{24} + 2 \\ \Rightarrow \quad p_P &= \tfrac{5}{24} + \tfrac{8}{24} = \tfrac{13}{24} = 0.542. \end{aligned}$$

Substitute $p_P = \frac{13}{24}$ and $p_R = \frac{10}{24}$ in (a):

$$\begin{aligned} p_S &= 0.3 \times \tfrac{13}{24} + 0.7 \times \tfrac{10}{24} \\ \Rightarrow \quad p_S &= \tfrac{109}{240} = 0.454. \end{aligned}$$

The final answer is:

$$p_S = \mathbb{P}\,(\text{visitor attends the Lecture} \,|\, \text{start at state } S) = \tfrac{109}{240} = 0.454.$$

(5)

Continued ...

(b) Define the following expectations:

$$\begin{aligned} m_S &= \mathbb{E}\,(\text{number of Rides}\,|\,\text{start at state } S) \\ m_P &= \mathbb{E}\,(\text{number of Rides}\,|\,\text{start at state } P) \\ m_R &= \mathbb{E}\,(\text{number of Rides}\,|\,\text{start at state } R) \end{aligned}$$

First-step analysis equations: we add 1 ride only on the **arrows leaving state *R***.

$$\begin{aligned} m_S &= 0.3m_P + 0.7m_R && \text{(a)} \\ m_P &= 0.4m_P + 0.3m_R && \text{(b)} \\ m_R &= 1 + 0.5m_R + 0.2m_P && \text{(c)} \quad \text{(Add 1 for this equation only.)} \end{aligned}$$

Substituting (b) into 2×(c) and rearranging:

$$m_R = 2 + 0.4\left(\tfrac{3}{6}\right)m_R \quad \Rightarrow \quad m_R = \tfrac{2}{0.8} = 2.5.$$

Thus from (b) and (a):

$$\begin{aligned} m_P &= \tfrac{3}{6}m_R = 1.25\,; \\ m_S &= 0.3 \times 1.25 + 0.7 \times 2.5 = 2.125. \end{aligned}$$

The final answer is:

$$m_S = \mathbb{E}\,(\text{number of Rides}\,|\,\text{start at state } S) = \tfrac{17}{8} = 2.125.$$

(4)

(c) We are told there are 12 people currently in state P, and 48 people in state R.

Each of the 12 people in state P eventually attend the lecture independently with probability $p_P = \frac{13}{24}$ from part (a). Thus the number of these 12 who attend the lecture is $X_1 \sim \text{Binomial}(12, \frac{13}{24})$ and the expected number is $\mathbb{E}(X_1) = 12 \times \frac{13}{24} = 6.5$.

Each of the 48 people in state R eventually attend the lecture independently with probability $p_R = \frac{10}{24}$ from part (a). Thus the number of these 48 who attend the lecture is $X_2 \sim \text{Binomial}(48, \frac{10}{24})$ and the expected number is $\mathbb{E}(X_2) = 48 \times \frac{10}{24} = 20$.

Overall, the expected number of people to attend the Lecture out of these 60 people is:

$$\mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 6.5 + 20 = 26.5.$$

(3)

Continued ...

2.(a) Consider:

$$Y \sim \text{Poisson}(\lambda)\,;$$
$$X \,|\, Y \sim \text{Poisson}(3 + Y)\,.$$

Then

$$\mathbb{E}(X \,|\, Y) = 3 + Y \quad \text{and} \quad \text{Var}(X \,|\, Y) = 3 + Y.$$

Thus:

$$\begin{aligned}
\mathbb{E}\,(X + Y \,|\, Y) &= \mathbb{E}\,(X \,|\, Y) + Y \quad \text{(treating } Y \text{ as a constant within the expectation)} \\
&= 3 + Y + Y \\
&= 3 + 2Y. \qquad (1)
\end{aligned}$$

Similarly:

$$\begin{aligned}
\text{Var}\,(X + Y \,|\, Y) &= \text{Var}\,(X \,|\, Y) \quad (Y \text{ is a constant within the variance, and constants vanish}) \\
&= 3 + Y. \qquad (2)
\end{aligned}$$

Using the Law of Total Expectation:

$$\begin{aligned}
\mathbb{E}(X + Y) &= \mathbb{E}_Y\,\{\mathbb{E}(X + Y \,|\, Y)\} \\
&= \mathbb{E}_Y\,\{3 + 2Y\} \qquad \text{using (1) above} \\
\therefore \quad \mathbb{E}\,(X + Y) &= 3 + 2\lambda.
\end{aligned}$$

Using the Law of Total Variance:

$$\begin{aligned}
\text{Var}(X + Y) &= \mathbb{E}_Y\,\{\text{Var}(X + Y \,|\, Y)\} + \text{Var}_Y\,\{\mathbb{E}(X + Y \,|\, Y)\} \\
&= \mathbb{E}_Y\,\{3 + Y\} + \text{Var}_Y\,\{3 + 2Y\} \qquad \text{using (1) and (2) above} \\
&= 3 + \lambda + 2^2 \times \lambda \\
\therefore \quad \text{Var}\,(X + Y) &= 3 + 5\lambda.
\end{aligned}$$

(5)

Continued ...

(b) Using probability as a conditional expectation (which is identical to the Partition Theorem for a discrete random variable Y):

$$\begin{aligned}
\mathbb{P}(X+Y=10) &= \sum_{y=0}^{10} \mathbb{P}(X+Y=10 \,|\, Y=y)\mathbb{P}(Y=y) \\
&\qquad \text{(note that } \mathbb{P}(X+Y=10|Y=y)=0 \text{ if } y>10, \\
&\qquad \text{so the sum only goes up to } y=10) \\
&= \sum_{y=0}^{10} \mathbb{P}(X=10-y \,|\, Y=y)\mathbb{P}(Y=y) \\
&= \sum_{y=0}^{10} \frac{(3+y)^{10-y}}{(10-y)!}\, e^{-(3+y)}\, \mathbb{P}(Y=y) \\
&\qquad \text{because } [X \,|\, Y=y] \sim \text{Poisson}(3+y) \\
&= \sum_{y=0}^{10} \frac{(3+y)^{10-y}}{(10-y)!}\, e^{-(3+y)} \times \frac{\lambda^y}{y!}\, e^{-\lambda} \\
\therefore \quad \mathbb{P}(X+Y=10) &= \sum_{y=0}^{10} \frac{\lambda^y (3+y)^{10-y}}{y!(10-y)!}\, e^{-(3+y+\lambda)}\,.
\end{aligned}$$

(3)

Total: (20)