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A STUDY OF EXPERTS' UNDERSTANDING OF ARITHMETIC MEAN

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ABSTRACT

Arithmetic mean is a concept which, although simple in form, is quite complicated to fully understand. The purpose of this study was to examine how experts understood arithmetic mean in order to provide articulations of their understandings. The articulation of conceptions of arithmetic mean could give a research-based foundation to future studies. Five participants, each with expertise with arithmetic mean, took part in task-based interviews to probe into how they were thinking of arithmetic mean. Analysis of the interviews led to an articulation of two distinct conceptions of arithmetic mean: understanding the algorithm for arithmetic mean and understanding arithmetic mean as a mathematical point of balance. The experts did not have a readily-available description of how these conceptions were connected. A follow-up interview was conducted with four of the participants to further clarify how they were thinking about *both* conceptions as arithmetic mean and how these conceptions might be connected. The analysis of the telephone interviews in conjunction with the analysis of the previous interviews led to a hypothesized connection between the conceptions through the use of leveling-off. This hypothesis lays a foundation for future studies of how children may develop a fuller and more connected understanding of arithmetic mean.

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Chapter 1

Purpose Statement

1.1 Purpose of the Study

The purpose of this study was to answer the question, “What are the goal understandings for teaching arithmetic mean?” That is, how do those who understand the arithmetic mean conceive of it? What are the critical components of an understanding of the arithmetic mean? Through the answers to these questions, hypotheses may be developed for instruction of arithmetic mean for understanding.

1.2 History of the Question

Researchers and teachers generally agree that a goal of mathematics education is that students understand the mathematics they are learning. Although there is usually not a clear consensus as to what is meant by “understanding,” it is, nevertheless, an end goal of much that has been and is being done in the realm of mathematics education. Over the years, researchers have tried to define what is meant by understanding. These definitions range from the philosophical definitions given by Sierpinska (1994), to the very pragmatic definitions given by Wiggins and McTighe (1998). What most of these definitions have in common is that they focus upon the action of understanding, and in doing so, focus on what is occurring in the “understand-er,” the person in whom understanding occurs. What they do not focus on is what Sierpinska has termed the “object of understanding,” that is, the mathematical (mental) structures that the learner constructs.

Despite the fact that focus of the treatises on understanding tend to overlook the object, mathematics education researchers have not ignored the object of understanding in their programs of study. As an example, Carpenter and Moser (1983) carefully described how students could understand the different structures in addition and subtraction problems. Steffe, Cobb and Wheatley (Cobb & Wheatley, 1988; Steffe, 1988) worked on describing what children understood about the concept of ten as it is situated in our numerical system's use of base 10. More recently, much research has been conducted to describe how students understand ratio and proportion (Hart, 1988; Heinz, 2000; Lamon, 1995; Lesh, Post & Behr, 1988; Quintero, 1981; Schwartz, 1988; Thompson, 1994, 2003; Tzur, 1999). In this study, the main focus was on articulating understandings of arithmetic mean.

1.3 Benefits of the Study

In education, there are benefits to knowing what it is that students need to understand because a direction for pedagogy can be established. Describing the understandings of arithmetic mean is beneficial to mathematics educators, curriculum developers and researchers because it gives an end goal for their work. By "understandings," I mean a set of researcher-made descriptions (second-order) of what appears to be understood by those who apparently understand the concept. Evidence of such understanding is typified in their facility in using the concept or connecting it with other mathematical concepts.

The benefits of articulating understandings are multi-faceted. First, there is a benefit to the mathematics education research community. Articulation gives a basis upon which research may be built. Researchers can conduct focused studies on how students construct understandings, aiding curriculum developers and teachers in knowing how to better teach for the goal of student understanding. The articulation of understandings can be refined as researchers develop more in-depth studies, and the connections between various understandings can be more clearly defined through on-going research. Second, the articulation of understandings benefits curriculum developers. Where, when and how concepts are placed in the curriculum could be impacted by this greater clarity as to what structures are needed for a fuller understanding. For example, because the algorithm for arithmetic mean includes division, arithmetic mean should not be taught before division is understood. Pedagogical interventions could be created to aid students in the building up of the articulated understandings. Third, teachers benefit from the articulation of understandings. It is generally accepted that teachers need to set learning goals and objectives. It is reasonable to assume that these goals should be in terms of students' understanding of mathematical concepts reflected in what the student can say and do. Thus, teachers benefit from having a clearer description of the understandings which are needed for particular mathematical concepts. The articulation of understandings could guide teachers to develop lessons designed for the purpose of student understanding of a concept instead of simply facility in using a particular algorithm.

This is not to say that the articulation of understandings of mathematical concepts will revolutionize the mathematics education world. As is posited by the group of researchers involved in writing *Adding It Up*, there is more to being "mathematically proficient" than simply having conceptual understanding of a mathematical concept (Kilpatrick, Swafford & Findell, 2001, p.116). However, a strengthening of students' conceptual understanding would certainly lead to greater mathematical proficiency.

1.4 Arithmetic Mean as the Focus

Recently, there has been an increase in emphasis on data analysis and statistics in U. S. mathematics. In 1999, Cerrito wrote:

> Statistical literacy is no longer a luxury; it is a necessity. A saying commonly attributed to W. Edwards Demming, the primary pioneer of quality control, proclaims, 'In God we trust; others must have data.' Numbers permeate society and are constantly referred to in political dialogue. Those with credentials that label them experts are free to expound on numbers without fear of challenge. (p.9)

In our democratic society statistical literacy must be developed in order to have an informed citizenry.

Mathematics educators and researchers support this view of statistical literacy as a necessary goal for education (Bakker, 2001; Gal, 1990; Konold & Higgins, 2003; Watson & Moritz, 2000). In response, several countries' standards-creating bodies, including the National Council of Teachers of Mathematics (NCTM) in the United States, have emphasized that students need to become more versed in data analysis and statistics by making probability and data analysis focused content strands throughout a child's mathematics education (Department for Education and Employment, 1999; Ministry of Education–New Zealand, 1996; Ministry of Education–Ontario, 2005; National Council of Teachers of Mathematics, 2000; Victorian Curriculum and Assessment Authority, 2005).

The arithmetic mean has typically been taught by having students learn the algorithm before it is taught for understanding (often as a "real-world" application of division). This may lead to understanding the concept as simply a computational act (Morrow & Kenney, 1998). It was the purpose of this study to take this concept of arithmetic mean, which seems to be simple from an algorithmic point of view, and look more carefully at the constituent or underlying understandings.

Current literature suggests that it is more beneficial to students if they understand arithmetic mean as a representative number that characterizes a set of data rather than as the result of an arithmetic operation on a set of numbers (Cortina, 2001; Gal, 1995, 1998; Gal et al., 1989, 1990; Garfield, 2000; Jones, 2000; Konold, 2003; McClain & Cobb, 2001; Schaeffer, 1998; Watson & Moritz, 1999a, 1999b, 2000; Zawojewski & Shaughnessy, 2000). This suggestion is consistent with the one laid out in *Principles and Standards for School Mathematics*:

> Students need to understand that the mean “evens out” or “balances” a set of data and that the median identifies the “middle” of a data set. They should compare the utility of the mean and the median as measures of center for different data sets. As several authors have noted (e.g., Uccellini [1996]; Konold [forthcoming]), students often fail to apprehend many subtle aspects of the mean as a measure of center. Thus, the teacher has an important role in providing experiences that help students construct a solid understanding of the mean and its relation to other measures of center. (National Council of Teachers of Mathematics, 2000, p. 250)

One focus of the current study was on how arithmetic mean as “balance” is understood in the context of data sets.

1.5 Theoretical Background

The construction of knowledge is a psychological phenomenon that occurs in the human. Constructivism is a philosophy that became a basis for a theory of learning with contributions coming from several researchers who have added to its development (Piaget, Vygotsky, Von Glaserfeld). The current study and its research was basically constructivist in nature, building upon the major premise that knowledge is not “passively received through the senses or by way of communication, but must be actively built up by the cognizing subject" (Von Glaserfeld, 1995, p. 18). This building up occurs in the learner through a process of assimilation and accommodation of understandings as described by Piaget (Singer & Revenson, 1996). That is, students learn by reflecting on what they have done. They then either organize what they have reflected upon by fitting it into an existing schema (assimilation) or by reorganizing their schema (accommodation) to take the new reflection into account. To develop instructional interventions that assist students in "building up" their understanding of a concept, it is useful to have a sense of what the students currently understand and a description of what the goal understanding might be. In this way, thoughtful instruction can be developed that allows students to reflect on the instruction in such a way that they build up from the current understanding to the goal understanding.

Because the study is based on constructivism, the articulation of understandings are my description, as a researcher, of what I perceive to be the critical concepts, or relationships between concepts, of others' constructed understanding of arithmetic mean. A goal of the articulation is that it be useful for mathematics education. That is, I do not suggest that the described understandings are a description of exactly how someone understands the arithmetic mean. Instead, they are an inferred description of what I consider a beneficial way of describing their understandings of arithmetic mean in order to be useful for mathematics education and mathematics education research.

1.6 Methodology

The current study was exploratory in nature. Only a small number of individuals were interviewed in the process of collecting data. Additionally, because the experts who were interviewed were likely to possess conceptions or ways of understanding arithmetic mean that were unknown to me before the start of the study, it was necessary to continually adjust how I examined their understandings. The structure of the methodology was developed in order to give as much information to explore the ways the experts were thinking about arithmetic mean.

To develop an articulation of understandings of arithmetic mean, task-based interviews were used to probe the nature of the understandings of several individuals. These individuals were chosen because of their likelihood of possessing sophisticated understandings of arithmetic mean. An initial interview was developed based upon possible understandings of arithmetic mean that were identified from existing research. Additionally, the tasks were designed to help answer questions raised through the review of pertinent literature. The tasks used were designed to be open-ended in order to encourage the person being interviewed to answer in ways that may not have been predicted by the researcher. After an interview schedule was created, three people were interviewed. An analysis of the responses in the interview was then conducted and the results were used to hypothesize and articulate what might be understandings of arithmetic mean. This analysis included selecting segments of the interview that suggested instances of competent behavior and then inferring conceptions from the experts' behaviors and explanations. The interview questions were then studied. Wording that was considered unclear by the researcher was adjusted and additional questions to probe newly hypothesized understandings were added. A subsequent interview was then created and conducted with four new participants. The analysis of this second phase of interviews was used to continue to hypothesize and articulate understandings. This analysis led to a more specific focus in the study. A telephone interview was then conducted with all previous participants in order to clarify understandings related to this more specific focus. The analysis of this new interview added to the overall articulation of understandings of arithmetic mean.

1.7 Conclusion

The study confirmed that the experts used two distinct conceptions of arithmetic mean. Each concept was carefully articulated and illustrated. How these conceptions were connected for the experts was then explored. The exploration into a possible connection generated a hypothesis for a way to connect the two conceptions.

Chapter 2

Review of the Literature

2.1 Statement of the Problem

Recently, there has been a national focus on improving statistics education.

> People depend on data to make intelligent decisions, yet the data they see are often tainted. So, what can be done? Part of the answer lies in education. Every high school graduate must be educated to be an intelligent consumer of data and to know enough about the production of data to form reasonable judgments about the value of data provided by others. (Scheaffer, Watkins, & Landwehr, 1998, p.4)

In the 2000 publication of *Principles and Standards for School Mathematics,* the National Council of Teachers emphasized data analysis and probability as a strand for teaching throughout Pre-K to Grade 12 mathematics (NCTM, 2000). The arithmetic mean is a fundamental concept in data analysis because of its use in inferential statistics and formulas (Leon & Zawojewski, 1991).

A central concept of statistics, the arithmetic mean, is typically taught in a procedural manner. E. George stated, “The arithmetic mean is rarely taught as a concept, but rather as the outcome of a computational procedure” (George, 1995, p.3). However, the arithmetic mean is a concept that is quite complex. Although the arithmetic mean is a fundamental concept, researchers agree that “the arithmetic mean is a mathematical object of unappreciated complexity (belied by the ‘simple’ algorithm for finding it)” (Mokros & Russell, 1992, p. 29). Since it is not well understood, and yet is a fundamental part of statistics education, articulation of concepts that underlie arithmetic mean is important.

A goal of mathematics education is that students learn to understand mathematical concepts. This will then add to the students' mathematical proficiency as defined by the authors of *Adding It Up* (NRC, 2001). Agreement is not present in the mathematics education community as to what constitutes understanding of a mathematical concept or what it is that should be understood in order to understand a particular mathematical concept. In this chapter, I will begin by reviewing the literature on mathematical understanding followed by the usefulness of articulating understandings of a mathematical concept. In the third section, I will review research on understanding the arithmetic mean.

2.2 Understanding

Throughout the history of education, researchers have defined understanding in order to clarify one goal of education and subsequently add power to the particular means of education that were being studied. For example, John Dewey wrote that to understand something "is to see it in its relations to other things: to note how it operates or functions, what consequences follow from it, what causes it" (Dewey, 1991, p. 137). Because of the rich history of defining understanding, a problem arises when writing or studying understanding. That is, there is a lack of agreement as to what is meant by "understanding" (Meel, 2003).

In searching for agreement, or at least a definition that can be agreed upon for the purpose of research, many mathematics education researchers have looked back at Skemp's seminal work on instrumental and relational understanding (1987). In his work, Skemp wrote that there are two types of concepts: primary concepts that are derived from our experiences and secondary concepts that are abstracted from primary concepts. These concepts are structured by the human mind into "schema." "To understand something means to assimilate it into an appropriate schema" (Skemp, 1987, p. 29). This definition coincides with the purpose of the current study as the articulation of conceptual understandings begin to name both the primary and the secondary concepts that are involved in understanding the mathematical concept. Skemp also distinguished types of understanding: instrumental, relational, and formal.

> Instrumental understanding is the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works.
>
> Relational understanding is the ability to deduce specific rules and procedures from more general mathematical relationships.
>
> Formal [= logical in my table] understanding is the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning. (Skemp, 1987, p. 166)

This structure for defining understanding focuses on what is occurring in the learner. In fact, Skemp argued in the second edition of his book that the "is" in the definition be replaced by "is evidenced by" to reflect the nature of the researcher's/teacher's attempt at assessing the type of understanding a learner possesses. The understandings, as defined for this study, are also a researcher's description of what might be occurring in an "understander," but the focus is on the object of that understanding.

Another critical work on understanding in mathematics is that of Sierpinska (1994). In her book, Sierpinska delved into the philosophical meaning of understanding specifically in the area of mathematics. The definition used in her book was borrowed from Ajdukiewicz's (1974) definition: "an act of mentally relating the object of understanding to another object" (Sierpinska, p. 28). This definition requires identifying the "understand-er," the "object of understanding" and the "basis for understanding." In this study, the purpose of articulating understandings was to define the object of understanding and possibly the basis for that understanding.

Wiggins and McTighe state that "to understand a topic or subject is to use knowledge and skill in sophisticated, flexible ways" (Wiggins & McTighe, p. 24). They suggest there are degrees of specificity and that topics need "uncoverage" to determine what is worth understanding. In one sense, "uncoverage" is the goal of articulating understandings: articulating the knowledge and skills that are to be used in sophisticated and flexible ways. Wiggins and McTighe also wrote about understanding as a process or a matter of degrees. They refer to Bloom (1956) who in effect, through his taxonomy, classified degrees of understanding. Bruner (1960) also considered understanding to be a process. These "process" views of understanding were synthesized by Perkins when he wrote, "In a phrase, understanding is the ability to think and act flexibly with what one knows" (Wiske, 1998, p.40). Each of these descriptions of understanding is pragmatic– being able to *use* what is understood is part of understanding. The benefit of articulating the understandings of a mathematical concept is that ***what it is*** that is to be used flexibly is identified.

In writing to teachers about learning theories, Bigge and Shermis (1999) defined understanding as follows:

> An understanding of a thing or process is its generalized meaning; that is, it is a tested generalized insight. Thus, it entails one's ability to use an object, fact, process, or idea in several or even many somewhat different situations. It is one's understandings that enable one to behave intelligently with foresight of consequences. (p. 85)

As is true of all of these definitions of understanding, this definition focuses on two aspects. First, there is something to be understood: termed in various ways as: knowledge, facts, ideas, processes, insight. Second, there is a process involved in the understanding: terms such as: relating, using flexibly, and connecting. Thus, both a static object and a dynamic process are involved in understanding. It is the object of understanding that was the focus of this study.

Several mathematics education researchers have gone beyond *defining* understanding to describing ways to consider the *process* of mathematical understanding in a learner (Hiebert & Carpenter, 1992; Pirie & Kieren, 1994). The views on understanding represent researcher-made organizational schema for studying and writing about the development and growth of student understanding. Because the reviewed literature focused on general mathematical understandings, the authors did not focus specifically on the object of understanding. However, the articulation of understanding of a mathematical concept integrates well with several of these descriptions.

As an example, Hiebert and Carpenter developed a framework for describing understanding that provides a tool for organizing articulated understandings. The authors' framework is based upon the belief that because mathematical ideas are abstract, representations are necessary to communicate those ideas. These representations can be made external (symbols, pictures, language) to communicate with others, but they start as internal representations. Internal representations allow the mind to operate on the abstract mathematical ideas. Drawing on cognitive science, the framework is based on two assumptions. "First, we assume some relationship exists between external and internal representations. Second, we assume that internal representations can be related or connected to one another in useful ways" (Hiebert & Carpenter, 1992, p. 66). The authors describe the connections with two different metaphors because of the different natures of mathematical concepts. The first metaphor is that of vertical hierarchies and focuses on the organization of mental representations as that of an overarching generalization (a "big" idea) with some representations being subsumed as details (sub-ideas) under these generalizations. The second metaphor is that of a web and focuses on connections of mental representations as a spider-web-like structure with threads of connections between the representations (p. 67).

Hiebert and Carpenter concluded their discussion of this framework by stating, "If students' and teachers' understandings are important, then research efforts should be directed toward describing and explaining these understandings" (Hiebert & Carpenter, 1992, p. 92). In the articulation of the understandings of arithmetic mean, some understandings are hierarchical; they are needed before other understandings. For example, it may be that understanding a distribution as an aggregate is an understanding that forms a basis for understanding arithmetic mean. Other understandings are web-like. It is possible the understandings of arithmetic mean as a point of balance has a web-like relation to understanding arithmetic mean as the product of a leveling-off.

Another view of understanding that has been used in mathematics education research was developed by Pirie and Kieren. The model was based upon a belief that "understanding a child's understanding brings about teaching to build that understanding . . . One might even argue that it occurs recursively; prior understanding, whether of the child or teacher, are [sic] a basis for new understanding" (Kieren, 1990, p. 191). The authors developed a model to illustrate the growth of student understanding. They used eight concentric circles to show that there is a hierarchy, but there is also a “folding back” or a re-entering of levels in the process of understanding. These eight levels were titled by the authors as: Primitive knowing, Image making, Image Having, Property Noticing, Formalising, Observing, Structuring, and Inventising. This model focuses on the process of understanding. Articulation of the understandings of a mathematical concept is compatible with this model. This study has the potential to specify primitive knowing, relevant images, properties to be noticed, etc., for arithmetic mean.

Despite any perceived difficulties and differences in describing or defining “understanding,” mathematics educators agree that it is, nonetheless, an important goal of education.[1] Understanding is a process, as the definitions and views of understanding presented suggest, but it also involves an “object” as Sierpinska termed it (1994). Meel wrote, “the development of understanding is a process of connecting representations to a structured and cohesive network. The connection process requires the recognition of relationships between the piece of knowledge and the elements of the network as well as the structure as a whole” (Meel, 2003, p. 135). It was the goal of this study to articulate the pieces of knowledge and the relationships between those pieces in order to identify what is needed for a learner to have understanding of arithmetic mean.

2.3 Usefulness of Describing Understandings

I have described how articulating understandings integrates with definitions and views of understanding. In this section, I develop an argument for why the articulation of

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[1] This is in spite of the fact that there is great disagreement as to how we can achieve this goal as educators.

understandings could be beneficial to mathematics education researchers, curriculum developers and teachers.

In reviewing literature, research that has been conducted specifically for the purpose of articulating the understandings of a particular mathematical concept was not found. Instead, what was found is research that probes into how students already understand a particular concept (Ball, 1990; George, 1995; Niemi, 1996; Russell & Mokros, 1996; Silver, 1983), what students do not understand about a particular concept (Cai, 1995, 1998; Clement, 2001; Heller, 1990; Pollatsek, Lima & Wells, 1981; Squire, 2003), or how a particular concept might develop in students (Cobb, 1988, 1999; Confrey, 1988; Heinz, 2000; McClain & Cobb, 2001; Watson & Moritz, 2000).

In writing on designing instruction, Dubinsky wrote of “genetic decomposition.” He described this as a “description, in terms of our theory, and based on empirical data, of the mathematics involved and how a subject might make the constructions that would lead to an understanding of it . . .” (1991, p. 96). Identifying the understanding of arithmetic mean is similar to genetic decomposition. Empirical data and thought experiments (Simon, 1995) on the mathematics involved were used to articulate the understandings. The activity of hypothesizing about these constructions naturally flows from the articulations. Dubinsky suggested that the genetic decomposition was a part of the process of designing instruction. It seems reasonable that an articulation of the understandings of a mathematical concept is a worthwhile endeavor for assisting teachers and curriculum developers. The articulation provides goals for the creation of activities.

The articulation of the understandings of arithmetic mean could also be useful for researchers. Von Glaserfeld wrote, “In order to investigate how children form the basic concepts on which arithmetic can be built, it is indispensable to have a fairly explicit model of what these concepts might be in the adult" (von Glaserfeld, 1995, p. 161). A growing trend in mathematics education research is design research. Design research can be described as a three-step, iterative process. First, a "thought experiment" is conducted (Gravemeijer, 1994; Simon, 1995) to envision how the interaction of teaching and learning will occur in the classroom. This thought experiment usually includes what is known from previous research and from the experience of the researchers. From this thought experiment, an instructional design is built. The second step is the implementation of the instructional design in the classroom. The researcher(s) audio and/or video record the class for use in the third step. In the third step the researcher(s) analyze the classroom interactions from the audio/video tape to make further conjectures about the instructional design and make changes to the instructional design that are deemed useful. "The feedback of practical experience into (new) thought experiments induces an iteration of development and research." (Gravemeijer, 1994, p. 449). By using design research, the complexities of educational research are acknowledged, considered in the design, and therefore apparent in the analysis.

The first step of design research—the thought experiment—can be enhanced by the articulation of understandings. In writing about the design of their teaching sequence, Gravemeijer and his colleagues wrote, "Firstly, we ask ourselves: What constitutes the new mathematical reality we want the students to construe, and what are the mathematical relations involved? Secondly, we ask ourselves: What is the overarching model, and what do the underlying inscriptions consist of?" (Gravemeijer, 2002, p. 3). Similarly, Hancock's research team wrote, "We began with an image of how data modeling should look, a rough analysis of its component skills and processes . . ." (Hancock, Kaput & Goldsmith, 1992, p. 337). By articulating the understandings of arithmetic mean, a basis is laid for future design research that can begin with a thought experiment that is based upon research.

The development of the articulations of understanding arithmetic mean coincides well with the current proposal by Clements for a *Curriculum Research Framework* (2007). This framework was proposed in order to meet the need for scientifically based research in education. Clements suggested that an a priori foundation for this research is identifying subject-matter content. He states, "concept and procedures of the domain should play a central role in the subject-matter domain per se (Tyler, 1949), build from the students' past and present experiences (Dewey, 1902/1976), and be generative in students' development of future understanding" (p. 40). The articulations of arithmetic mean from this study were developed to be generative in nature.

2.4 Research on Arithmetic Mean

In the 1980's, researchers began to study school students' understanding of arithmetic mean as mathematics educators moved statistics education into primary and secondary education. Pollatsek, Lima & Wells (1981) used college students as subjects in their initial research. They concluded that the students viewed the mean as a computational act (add everything up and divide by number of elements) rather than a conceptual act. The authors studied the utility of using a balance model of arithmetic mean in developing student understanding of arithmetic mean. It was hypothesized that connecting what students knew about mean prior to the study (which seemed to be limited to an algorithm) with understandings of weights and deviations similar to balance would help students construct a more connected understanding of the arithmetic mean. The conclusion of this particular study was that it was not helpful because it replaced one set of poorly understood ideas with another. Mokros and Russell (1992) also described that for them a powerful understanding of mean is that of a mathematical point of balance. Unfortunately, the teaching interventions used in their research did not seem to increase their students' understanding of mean as a mathematical point of balance either.

Piaget and Inhelder (1958) and Siegler and Chen (1976, 1998) conducted studies on how children develop an understanding of balance. In the earliest study, the conclusion was that children could not fully develop an understanding of balance until they had reached a formal operational level of thinking. Siegler's studies suggested that understanding balance could be described in four stages by four "rules" that children use when trying to solve tasks involving balance. Full understanding was not achieved, according to Seigler, until the child arrived at rule IV. He concluded that earlier studies as well as his own study suggest that "full understanding of balance scale problems grows slowly, remaining below 50% through age 17-years (Jackson, 1965; Lee, 1971; Lovell, 1961)" (Siegler, 1976, p. 488).

Because the early research conducted on understanding balance suggests that it is a slow developing concept, it is not surprising that the studies on using balance did not appear to improve students' understanding of arithmetic mean. The possibility that students did not initially understand balance could have confounded the impact of the intervention on the students' understanding of arithmetic mean.

In 1988, Strauss and Bichler described what they viewed as understanding of the arithmetic mean. In their description they suggested seven properties of mean that should be understood:

> A: The average is located between the extreme value . . .
>
> B: The sum of the deviations from the average is zero . . .
>
> C: The average is influenced by values other than the average . . .
>
> D: The average does not necessarily equal one of the values that was summed . . .
>
> E: The average can be a fraction that has no counterpart in physical reality . . .
>
> F: When one calculates the average, a value of zero, if it appears, must be taken into account . . .
>
> G: The average value is representative of the values that were averaged. (Strauss & Bichler, 1988, pp. 65-66)

The seven properties appear to be a list of characteristics of arithmetic mean to be known rather than a description of understandings. In a full articulation of understandings of arithmetic mean, these seven properties should be natural consequences of that understanding.

The description of the seven characteristics was used as a starting point for several other research studies on the understanding of arithmetic mean. These studies examined whether or not students exhibited understanding of the characteristics. Some of the studies focused on middle school students’ understandings (Cai, 1998; Cai & Moyer, 1995; Mooney, 2002; George, 1995), while others focused on the understandings of elementary students (G. Brousseau, N. Brousseau, & Warfield, 2002; Russell & Mokros, 1996; Watson & Moritz, 1999). These studies indicated that it was common for students to not understand the representative nature of the arithmetic mean (item "G" from Strauss and Bichler's list).

Studies by Gal et al. focused on students' statistical reasoning (1989, 1990). The results suggested that students could know the algorithm for the arithmetic mean yet not recognize the need to use the arithmetic mean in a given context (see also: Cai & Moyer, 1995; Goodchild, 1988). This shortcoming was particularly evident when students were comparing data sets of unequal size and the use of the arithmetic mean was a reasonable choice. Gal and his colleagues suggested, "It is perhaps understanding of what an average *is*, what it stands for, that is the first hurdle which some of the children fail to pass" (Gal et. al, 1990, p. 9).

2.5 Identifying Understandings of Arithmetic Mean

In several studies the concept of measures of center were situated within broader studies on the topic of data analysis. These research studies suggested that the understandings of arithmetic mean are situated in the larger understandings of data analysis and distributions. Gal et al. (1995) gave a clear explanation for why situating the study of understanding of arithmetic mean within the understandings of data analysis and distribution was so vital. Using the metaphor of a painter knowing the best usages of a particular paintbrush rather than just the physical properties of that brush, they stated that we must teach *averaging* rather than knowledge of averages. That is, understanding the arithmetic mean requires viewing it as a useful tool in a meaningful context and understanding when it is genuinely needed. They suggested that students must know that the average is a tool and as a basic requirement should understand, "(1) What the average can be used for; (2) In what ways is it different than, or similar to, other tools; (3) Under what conditions it makes sense to use an average, and why; and (4) What might happen if an average is used when in fact it shouldn't be, or vice versa" (1995, p. 2).

It is reasonable to conclude that understandings of arithmetic mean must be grounded in understandings of distribution and would include aspects of understanding the usefulness of the arithmetic mean and its value in comparing distributions. Situating the arithmetic mean within the larger understanding of distribution is similar to the use of the terms "quantity" and "quantification" introduced by Thompson in his writing on ratio (1994). Thompson defined quantities as

> . . . conceptual entities. They exist in people's conceptions of situations. A person is thinking of a quantity when he or he conceives a quality of an object in such a way that this conception entails the quality's measurability. A quantity is schematic: It is composed of an object, a quality of the object, an appropriate unit or dimension, and a process by which to assign a numerical value to the quality. (1994, p. 184)

He contrasts this with his definition of quantification, "a process by which one assigns numerical values to qualities. That is, quantification is a process of direct or indirect measurement" (pp. 184-185). Gal's suggestion was that learners need to understand the quantity of arithmetic mean rather than just the quantification. Much of the current research on students' understanding of arithmetic mean has been conducted within studies on students' understandings of distributions (Bakker & Gravemeijer, in press; Bakker, 2003; McClain & Cobb, 2001). These studies include understanding arithmetic mean as just one part of understanding distribution. They focus on the quantity of arithmetic mean.

Other researchers have suggested that students need to view the distribution as a collection of individual scores but the arithmetic mean as a representative of that distribution. Goodchild wrote, “The idea of representativeness, [sic] and expectation need to be emphasised. A feeling for stochastic processes and the resulting distributions needs to be imparted together with an understanding that a stated average is a *measure* of a distribution” (Goodchild, 1988). More recently, Bakker, in writing about the concept of distribution wrote, “To move from a case-oriented view to an aggregate view on data, students need to develop a conceptual structure with which they can conceive data sets as aggregates. The concept of distribution is such a structure (Cobb, 1999; Gravemeijer, 1999)” (2003, p. 4). It has been suggested that “students will have to reorganize their thinking to be able to see data as possible values of a variable” (Hancock, Kaput & Goldsmith, 1992, p. 4) in order to develop this view. The value and purpose of using the arithmetic mean is lost if one is unable to view the arithmetic mean as representative of a distribution.

Researchers at Vanderbilt University have conducted several teaching experiments on data analysis and have included in their work developing an understanding of the arithmetic mean. McClain and Cobb (2001) hypothesized that multiplicative reasoning would be needed to support students’ understandings of data and distribution. Because the published work of their teaching experiment was focused on using micro-computer tools and developing socio-mathematical norms for studying students’ growing understandings of statistical data analysis, not on the understandings themselves, I contacted the research group for more information. Jose Cortina had worked on identifying understandings of arithmetic mean for use in the teaching experiment. Most of his work has not been published and much of it was based on other unpublished work by Thompson. In his published work, arithmetic mean is viewed as “a ratio that measures *group performance relative to the number of contributors in the group*" (Cortina, 2001). In the unpublished work, further development was done to describe the arithmetic mean as a ratio and describe ways to support this particular understanding. Understanding arithmetic mean as a ratio addressed the multiplicative relation between total accumulation of a group and group size. They concluded that students need to first conceive of group performance as a *measurable* attribute. The students could then conceive of mean as an appropriate measure of this attribute of group performance (Cortina et al., 1999 p. 7).

A similar result came from the work of Gal et al. The children interviewed in their studies did not have the appropriate proportional reasoning skills necessary for creating accurate responses to questions relating to the arithmetic mean. Both the Vanderbilt study and the Gal et al. studies refer to proportional reasoning because arithmetic mean is a tool for comparing data sets of different sizes. "Such comparisons cannot be easily made only on the basis of reference to absolute numbers of quantities. Rather, one needs to reason proportionally and use statistical tools or heuristics that overcome the inequality in dataset size" (Gal, 1990, p. 3). Cortina and his colleagues (2001) suggested that in order for students to see the arithmetic mean as a sensible choice, they need to have an "underlying sense of 'individual contributions to the group attribute.' That is, one must imagine that in a group each individual performs and that each performance contributes to the group performance" (p. 2). Cortina's work suggests that a view of the distribution as an aggregate, although necessary, may not be sufficient. It is also important to understand the arithmetic mean as a ratio that can be used to relate the individual contribution to the group's overall performance.

There are several ways in which ratio has been described in mathematics education research that are useful in the description of arithmetic mean as a ratio. First, the arithmetic mean is a "between" ratio. Mathematics educators agree that ratios can be thought of as either "within" ratios or "between" ratios. "Freudenthal (1973, 1978) classified a ratio as a within (or internal) ratio if it compared two quantities belonging to the same system which he defined as a measure space, for example, (5 flowers): (7 flowers). He classified a ratio that compared two quantities as different measure spaces as a between (or external) ratio, for example, (8 flowers): (2 vases) (Romberg et al., 1988)" (Heinz, 2000, p. 30, underline in original). The arithmetic mean can be classified as a "between" ratio because of the differing measure spaces used to create the mean, that of the individual's performance and the number of individuals represented in the group.

Second, the arithmetic mean can also be thought of as a special kind of between ratio, that of a "unit factor" as Kaput and West used the term (1994). In a unit factor, one quantity is considered as a unit and the other quantity is adjusted proportionally to match the unit of the first quantity. This approach to thinking of ratio can be used in understanding arithmetic mean. Each data point (individual's performance) is considered the unit and arithmetic mean is an adjustment of each data point representing how much each unit would have if the values of the data were evenly distributed among all the data points. This is similar to the "leveling-off" strategy used in many instructional interventions (George, 1995; Lappan, 1997; Konold, 2003; Stein & Lane, 1996). In this strategy of finding the arithmetic mean, students are instructed to think of how much value each data point would have if every data point had the same weight. This focuses the students on the contribution that each data point's weight makes to the overall group's performance. In his more recent writing, Cortina has called this a “normalized ratio,” recognizing that all normalized ratios are not arithmetic mean (Cortina, 2002).

Understanding of “per one” unit factor may also not be sufficient to fully understand arithmetic mean. The arithmetic mean may also be thought of as a “ratio as measure.” Simon and Blume (1994) defined ratio-as-measure as “the ability to identify a ratio as the appropriate measure of a given attribute” (p. 184). Their study uncovered a critical link between *mathematization* and understanding *ratio-as-measure*. “*Mathematization* is the process of identifying and representing the quantitative relations (Thompson, in press) of a real-world situation. *Real-world* here denotes a context that includes more than quantitative information” (p. 190, footnote). Viewing the arithmetic mean as a ratio-as-measure thus relates to what Cortina, Gal, and other researchers of the arithmetic mean have stated: it is necessary to see the usefulness of the arithmetic mean in the context in which it is most often used and it is critical to view it as a measure of an aggregate, the distribution. However, understanding ratio-as-measure adds the necessity of viewing arithmetic mean, a ratio, as a reasonable measure for representing the distribution. This requires understanding distribution as a collection of individual points, but also an entity that can be represented by one ratio. Thus the data points and their weights are in a multiplicative relationship to one another.

These studies on understanding arithmetic mean demonstrate that there are some uses of the arithmetic mean that would display how a student understands it (Strauss & Bichler, 1988), and that there are also some underlying structures that are related to the understandings of arithmetic mean. Because several unique understandings have been identified by the review of literature, a general question is raised: How are the identified understandings related or connected? The research reviewed also leads to several specific questions about the understandings of arithmetic mean.

2.6 Questions Raised by Literature Review

The concept of balance was highlighted by Pollatsek, Lima & Wells (1981) and Mokros and Russell (1992). The researchers suggested that there is a connection between the conceptual understandings of balance and the conceptual understandings of mean. However, when students had one set of the conceptual understandings before the other set of understandings, (such as understanding balance before arithmetic mean), their learning of the new concept was not enhanced. Yet the researchers were convinced that thinking of arithmetic mean as a point of balance was useful because it highlighted the mean as a measure of center. Additionally, much of current curriculum and research programs involve using an understanding of a leveling-off process to help students develop understanding of arithmetic mean. Both of these ways of thinking of arithmetic mean appear to be important. What is the connection between these two conceptions? What of the understanding of balance is called upon in the use of leveling off? What in the understanding of leveling off is called upon in the use of balance? What is the nature of the connection between these two understandings? How do experts conceive of arithmetic mean as a point of balance? What is the nature of the connections between balance and the algorithm?

Arithmetic mean is a ratio as was described by Cortina et al. What is not clear is how understanding of ratio is integrated with other understandings of arithmetic mean. What characteristics of understanding ratio are present in the understandings of mean as a point of balance? What is the nature of the connection between these understandings?

What characteristics of understanding ratio are present in the understanding of mean as a leveling-off process? What is the nature of the connections between these understandings?

The research also highlighted that an understanding of distributions has a great impact on the understanding of mean. Because the arithmetic mean *is* a measure that represents the distribution it is important to have a concept of distribution as a measurable entity. The understanding of a distribution being measurable includes understanding that although there are individual contributions, the distribution can be thought of as one unit. It also includes understanding the individual contribution's impact on the group's attributes. Why is the arithmetic mean a reasonable choice for a measure of the distribution? What characteristics of understanding mean (as balance, as leveling-off, as ratio) are connected to understanding the distribution as a measurable entity? What is the nature of those connections?

Chapter 3

Methodology

3.1 Statement of Methodology

The goal of the current study is to identify and articulate the understandings of arithmetic mean. To develop this articulation through research, it is necessary to use a methodology that is exploratory in nature. Both the general methodology of Grounded Theory and the specific methodology of Phenomenography have provided a foundation for the methodology constructed for use in the current study.

"Grounded theory is a *general methodology* for developing theory that is grounded in data systematically gathered and analyzed" (Denzin & Lincoln, 1988, p. 158). Grounded theory is designed to build theory. During analysis, hypotheses are constantly compared with the data collected in order to develop theory (at any level of theory building). A goal of the current study is to contribute to mathematics educator's understanding of arithmetic mean for the purpose of building a theoretical approach to the teaching of this concept.

"Phenomenography" is a term coined in 1972 by a group of educational researchers in the Department of Education at the University of Gothenburg, Sweden (Marton, 1988). The group studied how people thought about different concepts (not all mathematical). They concluded that "*each phenomenon, concept or principle can be understood in a limited number of qualitatively different ways*" (Marton, 1988, p. 143, Italics in the original). Phenomenography was then described as "a research method for mapping the qualitatively different ways in which people experience, conceptualize, perceive, and understand various aspects of, and phenomena in, the world around them" (p. 144). The phenomenographer is more interested in the content of thinking rather than the process. It is the shared interest in the content of thinking (object of understanding) that is borrowed from the methodology.

Marton and his colleagues mapped the qualitatively different ways Dutch students understood the physics concept of force. They produced a mapping of students' qualitatively different ways of understanding the concept. These different ways included understandings considered incorrect, naïve, or correct. In the current study, the focus of the mapping of understandings is with individuals who are considered to already possess a correct and more developed understanding of arithmetic mean. In this way, an articulation of goal understandings for students can be developed from analysis of experts' understandings. Most phenomenographers have focused on *all* of the qualitatively different ways of understanding a concept (both correct and incorrect); however, they recognize that there is a desired understanding (or goal understanding).

> For any particular text, the logical relations amongst the different categories of outcome were taken to define what Marton and Dahlgren (1976) described as an "outcome space." Particular outcomes could be regarded as being more appropriate or desirable than others, insofar as they bore a closer relationship to the author's original conception (Marton, 1976). More generally, one could say that some ways of experiencing a phenomenon were better than others to the extent that they were more "efficient" in terms of some given criterion: for instance, it could be said to be more efficient in terms of using and developing arithmetic skills to appreciate that numerical addition was a commutative relation, so that 2 + 7 = 7 + 2 (Marton, 1994). (Richardson, 1999, pp. 54-55)

In this study, my focus is not on all of the understandings. Rather, I focus on experts' understandings in order to hypothesize possible goal understandings for future studies on the learning of arithmetic mean. This study was different from the expert-novice literature in that the focus was only on descriptions of the experts' understandings, not on what was not understood or differences in understandings from novices.

The sources of data used in the current study are similar to sources used in phenomenography. The task-based interview was used to collect information about subjects' understandings of arithmetic mean. The subjects involved were selected, based upon their educational background, to represent those who understand arithmetic mean. The articulation of the understandings of arithmetic mean is derived from those whose understandings represent the goal understanding for arithmetic mean.

3.2 Description of Methodology

The task-based interview setting is used as a means of gathering information about each subject's understanding. Piaget (1929) wrote of using this technique as a method between pure observation and the clinical method, or test. The goal of a task-based interview is to "understand students' current reasoning patterns without attempting to change them" (Engelhardt, Corpuz, Ozimek, & Rebello, 2004, p. 1). Goldin (2000) gave an in-depth description of the task-based interview. He stated that the task-based interview involves at minimum three components: the problem solver, the clinician or interviewer and the tasks. The tasks are introduced by the clinician to the problem solver in a pre-planned way and "explicit provision is made too for contingencies that may occur as the interview proceeds, possibly by means of branching sequences of heuristic questions, hints, related problems in sequence, retrospective questions, or other interventions by the clinician" (Goldin, 2000, p. 519). Because of the pre-planned intervention and the plan for contingencies, the task-based interview is often called "semi-structured."

Goldin also suggested that the chosen tasks use "rich representational structures." In the task-based interview, the behavior of the subject being interviewed in relation to the structured interventions is observed. Rich representational structures (e.g., computer environments, use of several mathematical representations of one concept) give the researcher a broader range from which to infer what the subjects are thinking or feeling as they work through the tasks. The tasks should encourage unconstrained problem solving as this gives the researcher more observable behavior to analyze. Additionally, the tasks should encourage a maximum of interaction with the external learning environment as this also maximizes observable behavior. In order for the tasks to provide this type of environment, the design of the task should allow the subject to go in directions the researcher may not have considered.

There are several advantages to using the task-based interview. Goldin states that the value of the task-based interview "lies in the fact that . . . [it provides] a structured mathematical environment that, to some extent, can be controlled" (Goldin, 2000, p. 520). Yet the flexibility of the interviewer also gives value to the interview as it provides for the unexpected responses. Clement suggested that task-based interviews "strengths, in comparison to nonclinical, data-gathering techniques, include the ability to collect and analyze data on mental processes at the level of a subject's authentic ideas and meaning, and to expose hidden structures and processes in the subject's thinking that could not be detected by less open-ended techniques" (Clement, 2000, p. 547). The task-based interview gives a place for the researcher to gain insight into how the subject understands the concept or insight for a way to model the subject's understanding. In writing about phenomenography, Marton states:

> The point of departure in phenomenography is always relational. We deal with the relation between the individual and some specified aspect of the world, or, to state it differently, we try to describe an aspect of the world as it appears to the individual. This means that we adopt an experiential, or what phenomenographers call a 'second-order' perspective (Marton, 1981). We do not try to describe things as they are, nor do we discuss whether or not things can be described "as they are"; rather, we try to characterize how things appear to people. (Marton, 1988, p. 146)

Similarly, this study characterizes how people who have more sophisticated understandings of arithmetic mean appear to understand arithmetic mean.

The nature of this exploratory study is different from many task-based interviews. The researcher typically understands the concepts being examined better than those being interviewed. In the current study, I may not have known as much about the concept of arithmetic mean as those being interviewed. In this sense, I had to "bootstrap" my understanding throughout the interview.

3.3 Description of Phases of Interviews

The current study presented a dilemma. Because I was trying to find out how experts understood arithmetic mean, I did not know a priori what problems would best reveal those understandings. Unlike studies of students' understandings where the researchers' understanding provides a frame, no such overarching frame was available. Therefore, the interview tasks and probes evolved from subject to subject and from one phase of the research to the next. Because I was trying to find examples of sophisticated understanding (and not trying to make claims about the population of experts) modifying the protocol was generative rather than problematic.

3.3.1 Phase One of Interview

The initial interview, called "Phase One," was based on current literature on understanding arithmetic mean and an articulation of my personal understanding of arithmetic mean (see Appendix A). After a pilot interview was conducted with a mathematics education professor to ensure the usefulness of the interview tasks, the interview was conducted with three participants. The interviews were audio and video recorded and then transcribed and annotated. The annotated transcripts were analyzed as outlined in Section 3.6 of this chapter. During the analysis of the Phase One interviews, articulation of the subject's understanding of arithmetic mean was developed. Only one of these three interviews provided useful data for the following reasons. First, not all those interviewed had a strong conceptual understanding of arithmetic mean. Second, the interviews often did not reveal the conceptions the participant was using.

The tasks used in the interview were assessed for their effectiveness in revealing participants' understandings. The analysis of understandings and the assessment of tasks were used to either keep the existing tasks with some adjustments made to how the task was presented, or they were used to create new tasks for the next iteration of the interview called Phase Two. The new tasks were created to enhance the effectiveness of the interview, allowing me to better develop an articulation of the subject's understanding of arithmetic mean. They also were designed to allow for the new participants to possibly articulate ideas not yet observed.

3.3.2 Phase Two of Interview

Using a new interview schedule, the Phase Two interviews were conducted with four new participants and were also audio- and video- recorded. Three participants in this phase were full professors at a large university. A full time statistician at a small, private university also participated. After the recordings were transcribed and annotated, the Phase Two interviews were analyzed in the manner of the Phase One interview.

Several iterations of describing the understandings of arithmetic mean and then critiquing those written descriptions were conducted. These iterations of analysis led me and my doctoral advisor to begin to pose new questions that were not answered by the current data. Other questions were not clearly answered in the data as well. During this second phase, the interview schedule had adjustments made as described in section 3.6. A third phase of interviewing was conducted in order to try to answer some of the questions from the Phase Two analysis.

3.3.3 Phase Three

A follow-up telephone interview was conducted with all of the participants from Phase One and Phase Two whose data were being used in the study except for the full time statistician. This telephone interview was developed based on a response by the statistician in the Phase Two interview that was critical in the final hypothesis. Because these data were from only one participant, it was necessary to collect more data to analyze and determine the efficacy of the hypothesis. The telephone interview focused on one question from the Phase Two interview (see Appendix B). Each participant was contacted by e-mail and a brief 10-15 minute interview was conducted over the telephone. The conversation was audio taped.

3.4 Description of Participants

The participants of this study were all invited to be a part of the research. Mathematics education, statistics, and mechanical engineering professors from a large research university as well as from a small, private university were invited to be interviewed. Additionally, graduate students in mathematics education from a large university were invited to be interviewed. The full professors were selected for participation based upon their history of use of the concept of arithmetic mean within their research agenda or course work. The graduate students were selected for invitation because of their involvement in a three-year long study on undergraduates' understanding of statistical concepts. In the midst of their research study, much discussion and thought were given to the understanding of arithmetic mean.

The set of participants in the current study represent a group of people who have need of understanding arithmetic mean, but for differing purposes. Because of the participants' active use of the arithmetic mean in education and research, it was assumed that their understanding of arithmetic mean was not simply a computational understanding, but also a conceptual understanding. They were considered by me and by my colleagues as "experts" in areas of statistical concepts.

Each participant was given a letter pseudonym in the study. The participants included are: B, a mathematics educator; C, a graduate student; J, a statistics educator; K, a statistician; and L, a mechanical engineer. In order to conceal the identity of the participants, the masculine form is used throughout the written descriptions, despite the fact that females were included in the study.

3.5 Phase One Interview Tasks

The Phase One interview was designed using current research literature on arithmetic mean and my articulated understanding of arithmetic mean. Additionally, the questions raised from the review of literature were used as a structural tool for the development of the tasks. Each task was designed to examine a way of understanding arithmetic mean that would help to answer the questions. The overall goal of designing the tasks was to provide a rich environment in which the participant being interviewed could use several qualitatively different ways of understanding arithmetic mean and also articulate why they chose to think of arithmetic mean in that manner. This allowed me to probe into the statements made by the participant in order to better imply how the participant understood arithmetic mean. The questions follow along with a "name" for the problem. This name is used throughout the analysis and discussion of the data.

Question 1, Part 1: "Histogram Problem"

Hand participant histogram on handout 1
Figure **3-1**

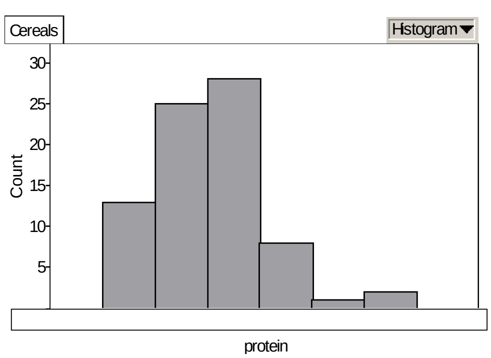

Figure **3-1**. Histogram Problem.

This data comes from the DASL website and is for free use. The histogram shown is data that is taken from the panel of 77 different breakfast cereals. The histogram shows the grams of protein in these cereals. These grams are integer amounts, but I have not shown how many grams. The count is the number of boxes of cereal that have a particular amount of grams of protein. Do you have any questions about reading this histogram? (Answer questions, if necessary)

ASK: *Can you find a point through which a vertical line would divide the graph into two pieces of equal area?* Be sure participant marks this point on the paper.

ASK: *Is there a statistical measure that this point is identified with?*

If participant answers "yes": *Which one?*

"Mean" *Would you draw a line where the median would be? Can you explain what you are doing to decide where the line belongs?*

Be sure to probe as to what characteristics of the histogram are being examined to make this decision. Probe into WHY the answer is given. *Would having the number of grams revealed help you at all? How?*

"Median" *Would you draw a line where the mean would be? Can you explain what you are doing to decide where the line belongs?*

Be sure to probe as to what characteristics of the histogram are being examined to make this decision. Probe into WHY the answer is given. *Would having the number of grams revealed help you at all? How?*

If participant answers "no": *What would this point tell you about the data?*

This question was obtained and adapted from an interview I had helped to conduct in a study aimed at examining pre-service teachers' understanding of measures of center. It was chosen as the initial question because it presents data in a non-standard form (the provided histogram is displayed in a manner inconsistent with the question of the average protein) and asks the participant to begin to think of arithmetic mean as it is compared to other measures of center. The design of the question is such that a glimpse into how the participant views the distribution as measurable can be examined as well as what understanding of arithmetic mean emerges first in this circumstance (balance, leveling-off). During the pilot interview, it was determined that by not giving the participant the integer amounts of protein, but simply calling them integer amounts, allowed me to probe deeper into the participant's understanding of arithmetic mean as a point of balance.

Question 1, Part 2: Histogram Problem

ASK*: On the computer I have a program called "Fathom". Do you know how to get Fathom to show the mean on that graph?*

If yes: *Please show me*

If no: *I can show you how. Go to the graph menu and then plot value. Now under function expand "statistical" and "one-variable". Double click on "mean". Now click OK.*

ASK*: Compare your answer to the one Fathom generated. Does Fathom's determination of the location of the mean make sense to you? How so? Or Why not?*

Possible answers and follow ups:

Skewness: *What about skewness makes the answers different? What if the data were skewed in the other direction? What if the data were not skewed?*

Found average of the range: *Can you show me what you mean using some numbers? Why would this be different from Fathom's answer?*

Did not take each height into account: *What would you have done differently to take the heights into account? What did those heights represent?*

ASK: *Moving as few points as possible, how could you alter the data set to make the median be to right of the mean? Why did you choose those points?*

In this part, the participant was asked to compare the answer given to a computer-generated "correct" answer. The follow-up questions are used to again probe what understanding of arithmetic mean appears to be the main understanding called upon in this non-standard situation. The question also allowed the participant to explain any inconsistencies, particularly if the first answer given was just an estimate.

Question 2: "Arithmetic Mean Definition"

Give participant a blank sheet of paper and a pen.

ASK: *If there were no such thing as arithmetic mean, would you invent it? Why?* After participant has finished, probe for factors he/she finds valuable and why.

ASK*:* *In (your area of expertise), is there a particular aspect of arithmetic mean that your students find difficult? Why do you think that is?* Probe carefully as to how they view the problem situation, what they might see as a remedy and why.

ASK: *Without using the algorithm, how would you define arithmetic mean? Why did you include _____ in your definition?*

These three questions were suggested to me by my advisor. They are placed second in the interview for a purpose. The first question allowed the participants to get their minds focused upon the aim of the study: discussing their understanding of arithmetic mean. However, it was a goal of the study to obtain candid answers to the three questions of Question 2 without undue influence from the rest of the interview questions. Thus, this was placed as the second question of the interview. The pilot interview suggested that these questions would provide information that could be used in the rest of the interview to examine relationships between different understandings of arithmetic mean. That is, the interview participant provided a definition, which appeared to include a primary way of understanding the arithmetic mean. Throughout the rest of the interview, I asked the participant to compare the work that was presently being done with the definitions provided earlier by this particular question.

Question 3: "Class Average Problem"

Handout with following situation:

In a certain class there are more than 20 and fewer than 40 students. On a recent test the average passing mark was 75. The average failing mark was 48 and the class average was 66. The teacher then raised every grade 5 points. As a result the average passing mark became 77.5 and the average failing mark became 45. If 65 is the established minimum for passing, how many students had their grades changed from failing to passing?

ASK: *Please read this situation. Do you have any questions about the situation? Would you please explain what you are doing as you solve this problem?*

As participant answers – probe as to why he/she is doing certain steps, what makes that important. At the end, probe to see if he/she has other ideas or another way of how to solve the problem that are more "proficient."

This question was originally created by Dr. Grayson Wheatley who graciously permitted me to include it in my study. There are several ways in which the problem can be solved, yet each way provided insight into understanding of the arithmetic mean. Of particular interest from this question was the participant's use or non-use of deviations from the arithmetic mean in solving this task.

Question 4: "Harmonic Mean Problem"

In the most recent issue of Mathematics Teacher, I read about an interesting mathematical relationship (Brown and Rizzardi, 2005). I'd like you to look at problems similar to the ones discussed and solve the problems as the authors suggest for the students to do.

Give Handout with the following Problems:

1. Two people drive from their hometown to a city that is 150 miles away. The first person is cautious and drives 50 mph. The second driver is a bit more reckless and drives 70 mph. Find the average speed.

2. My sister and I drove across the country and decided we would share the driving time equally. Every four hours we would stop to buy gas and coffee and trade driving. My sister started and was very excited, she drove 70 mph for the first four hours. When it was my turn we were headed up a lot of hills and I drove 50 mph for my four hours. What was the average speed for the eight hours?

3. Pat was training for a long run and decided to incorporate a particular training segment into his workout. He would run up a 1/2 mile long hill and then run down the hill for that segment. Running uphill, he ran at a pace of 10 minutes per mile. Running downhill, he ran at a pace of 8 minutes per mile. What was the average speed of his running during that segment of his workout?

4. The math education group decided to raise some money for charity by holding an "endurance" run. Two people would run for the first four hours, another two people for the next four hours and so on. During the first four hours, the two runners decided to trade off running every hour. Runner 1 ran one hour at the pace of 10 minutes per mile. Runner 2 then took over and ran at a pace of 8 minutes per mile for the next hour. What was the runners' average speed at the end of the first two hours?

ASK: *(As participants answer) Why did you choose to solve it that way? What cues were you using to help you?*

There is a possibility that harmonic mean will be used – if so, the next question can be skipped and the probing used later can be conducted here.

In the article, the authors wrote that you could use harmonic mean to find some of these averages. Are you familiar with the harmonic mean?

If NOT*: Write equation for harmonic mean on paper for participant to read and analyze.*

If SO*: continue.*

Here is the chart the authors developed to show when you could use harmonic mean and when you should use arithmetic mean.

Give Handout:

	Equal Distance	*Equal Time*
Distance – Oriented: *<u>Total Distance</u>* *Total Time*	*Harmonic Mean*	*Arithmetic Mean*
Time-Oriented: *<u>Total Time</u>* *Total Distance*	*Arithmetic Mean*	*Harmonic Mean*

ASK: *Do you understand this table? Can you relate each of the four problems given to one section of the table? How?* Once you are sure they have understanding of the situation, continue.

ASK: *The authors write that this relationship is true, but they do not explain WHY it is true. Could you explain why this particular relationship holds without using algebra?* PROBE

The first part of the question required the participant to find the average of rates. The result from the pilot interview suggested that this was a more difficult task for the participant than I had originally hypothesized. The questions did have the potential for revealing connections between the procedural understanding of arithmetic mean and the conceptual understanding. In the second part of the question, the participant was asked about the connection between the arithmetic mean and the harmonic mean. I initially did not know what the connection was on a conceptual level. I could only make the connection on a procedural (algebraic-manipulation) level. It was an aim to discover the connection and in the process, learn more of how experts understand arithmetic mean as it relates to other means.

Question 5: "Block Problem"

Give the participant handout with the drawing of Unifix cubes

Figure **3-2**

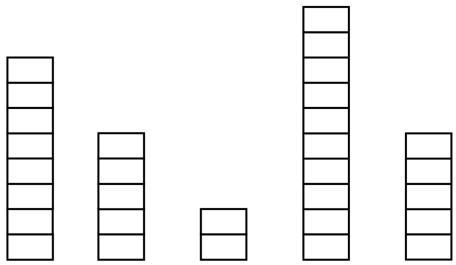

Figure **3-2**. Block Problem – Phase 1.

ASK: *Without using the algorithm, can you show me how you could find the arithmetic mean of the five numbers represented here?*

If participant uses "leveling-off" strategy: *Why does moving the blocks give the arithmetic mean? How is arithmetic mean shown in the blocks? When we find arithmetic mean, we get one number, why do all of the towers show the arithmetic mean?*

If participant uses another strategy: *What about your understanding of mean led you to use this strategy? How is the mean represented in what you've done?* Other probes:

ASK: *Is there another way you could find the mean without using the algorithm.* (If participant cannot think of way) *What if you didn't have the blocks, but just the numbers – how might you find the mean without the algorithm?* PROBE for why the strategy was chosen and how they understand what they are doing.

ASK*: How are your two strategies similar? How are they different? What in the given situation led you to choose the first strategy?*

The last question of the interview was written as an optional question. Several of the previous questions had the potential for giving me insight into how the participant related the understanding of arithmetic mean as a leveling-off process and the understanding of arithmetic mean as a balance point. However, it was not a guarantee that the participant would actually use these understandings earlier. Question 5 was added to provide an opportunity for me to examine the relationship between these two understandings. After the pilot interview, I was given another option for presenting the data for this question by the videographer (also a mathematics education graduate student). This new way of presenting the data removed the unit-based nature of the data and placed the problem in a less "contrived" situation. The same probing questions were used with the new situation:

Question 5: Revised "Building Height Problem"

While visiting a demolition site, you happened to look at a cityscape through a hole in the wall. You can see six buildings, and you know a seventh building is located to the right of the six in view. A construction worker gives you the average height of the buildings. How can you use the given information to determine the height of remaining building?

Figure 3-3

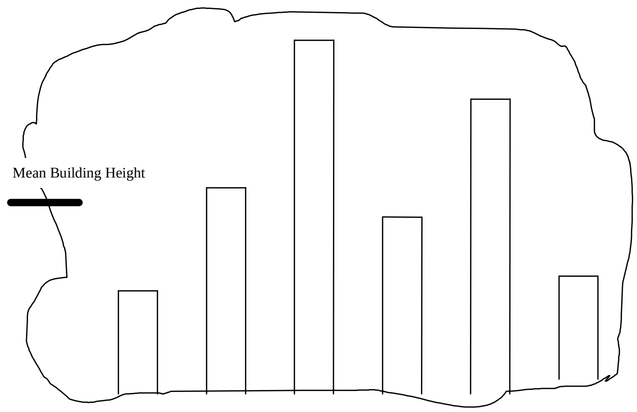

Figure **3-3**. Building Height Problem – Phase 1.

The new presentation of the data took away the possibility of the participant using solely a procedural, numeric approach, although the possibility for use of a procedure still remained.

As stated earlier, the phase one interview had its shortcomings both in the design of questions and in how I conducted the interview as a novice interviewer. Only one interview of the three provided useable data. Thus, the second phase interview was created.

3.6 Phase Two Interview Tasks

The second interview schedule was modified from the first interview.

Question 1: Histogram Problem

The first question was the Histogram Problem and remained the same in nature with the exception of a dotted red line placed on the histogram. The participant was told that this line divided the area in half. This reduced the time used in the previous interviews to find this point. Because the interview was about arithmetic mean, all of the labor done to find the point did not reveal much about a participant's conceptions that could not be revealed in other tasks.

Figure **3-4**

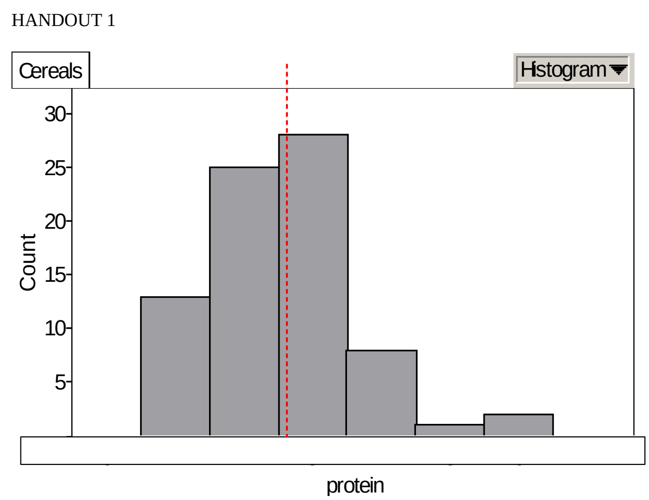

Figure **3-4**. Histogram Problem – Phase 2.

Question 2: Arithmetic Mean Definition

This question remained the same as it was useful in the Phase One interviews.

Question 3: Class Average Problem – Revised

The Class Average problem underwent a major change. In the Phase One interviews, algebra was relied upon heavily to solve the problem, and this masked any understanding the participant might have. Additionally, the complexity of the problem, with changing grades and having essentially six different sets of data, also masked potential understanding. Thus, attempts at having the participants accomplish the task actually stymied revealing possible understandings. After I met with my advisor to discuss this question, I took advantage of a local mathematics education research conference to pilot a new version of this task. The group that tried the newer version made many statements that revealed how they were conceiving of arithmetic mean. This encouragement led to the following changes to Question 3 which I entitle: “Class Average Problem – revised”

Handout with the following question: *A class of students took a test. The class average on the test was 68. The average grade of the students who passed was 80 and the average grade of the students who failed was 64. What percentage of the class passed?*

ASK: *Please read this situation. Do you have any questions about the situation? Without using algebra, would you solve this problem and explain to me how you are solving it?*

As participant answers, probe as to why he/she is doing certain steps and what makes that important. At the end, probe to see if he/she has other ideas or another way of how to solve the problem that are more proficient.

The Harmonic Mean Problem from Phase One was removed completely from the interview. The participants did not know enough about the harmonic mean to make the question useful. It was a very time consuming question that resulted in nothing useful for articulating the understanding of arithmetic mean. Question 5 then became two questions rather than one, and a new representation was brought in.

Question 4: “Building Height Problem”

Handout: Suppose the following picture is depicting building heights. What would the average height of the six buildings be? PROBE*: How are you deciding on that height? Are you sure that it is accurate? Why?*

Figure **3-5**

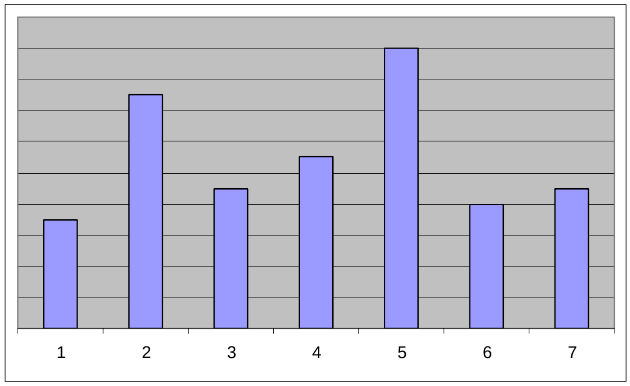

Figure **3-5**. Building Height Problem – Phase 2.

PROBE: *How does this relate to the definition of mean you gave me earlier? In the way you did this, where are you using ____ that you used in the earlier work with arithmetic mean?*

The representation was changed in order to give the participant a ready unit with which to measure. In the first phase of interviews the participants had used things like a pen or their own finger to measure. The horizontal lines reduced the complexity of the work in order to allow more questioning to focus on the understanding rather than on what was being done.

Question 5: "Dot Plot Problem"

Give participant the handout of distribution above a horizontal line (dot plot).

***Figure* 3-6**

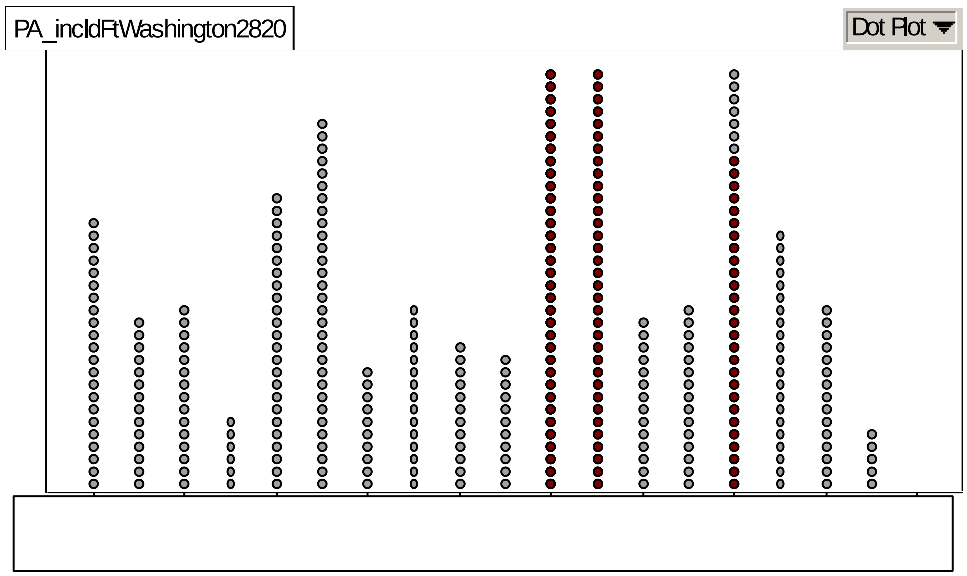

Figure **3-6**. Dot Plot Problem.

ASK: *This is a distribution of a large amount of data. The mean of the data is shown. This was created in Fathom, so the red dots mean that there is more than one data point being represented by the dot. If I added a data point here (mark 2 units to the right), how would that affect the mean?*

Probe: *How do you know the data point will move the mean in that direction? Do you know by how much? Why or Why not?*

ASK: *Now suppose that instead of the data point being added here (2 units to the right), it was added here (mark 4 units to the right). Now, how would that affect the mean?*

Probe: How do you know it will be impacted in that direction? Do you know by how much it will change? Do you know by how much in comparison to if it were this other point (2 units to the right)? How do you know that?

ASK: *In the above problem you used ______ to find the mean, but here you used _______ to change the mean. According to you, these both represent the mean. How then, are they related?*

These two questions allowed the interviewer to try to get the participant to articulate connections between leveling-off and balance. It also provided a scenario (in Question 5) that made the participant consider the nature of arithmetic mean as a representative ratio describing the aggregate (distribution). The Phase One interview did not seem to have any tasks that would address this vital issue.

Question 6: "Teacher Grading Problem"

A final question was added to the interview based on a Phase One interview response. A participant was responding to the question of what problems his students seemed to have. Instead of discussing his students, he discussed other teachers and the struggles they had with weighting grades. I created a problem in which a teacher averages grades two ways. I hoped that the question would help the participants to articulate connections between balance and ratio.

Give participant the copy of the excel file as shown.

Figure 3-7

HW-9	HW-5	Quiz -16	HW-5	Test - 42	HW-8	Quiz -20	Final - 63	Total - 168	GRADE
8	5	15	4	40	8	18	58	156	92.86
7	5	14	5	38	7	16	55	147	87.50
9	5	13	5	41	8	18	50	149	88.69
5	4	11	3	33	8	16	62	142	84.52

HW	HW	Quiz	HW	Test	HW	Quiz	Final	GRADE
88.89	100	93.75	80	95.24	100	90	92.06	92.49
77.78	100	87.50	100	90.48	87.5	80	87.30	88.82
100.00	100	81.25	100	97.62	100	90	79.37	93.53
55.56	80	68.75	60	78.57	100	80	98.41	77.66

Figure **3-7**. Teacher Grading Problem.

ASK: The top file shows the grade by points earned out of points possible and the final grade was figured from total points earned out of total points possible. In the lower file, each grade the students earned was converted to a percentage and the final grade was figured by finding the average percent grade for the eight graded assignments. Why do Students 3 and 4 have such different grades?

PROBE: *How do you know this is the cause of the difference? What are you using to help you make this determination?*

By focusing the participant's attention on the differences, the researcher desired to have the participant articulate how weighting affected the balance and how that was connected to distance from the average.

A need for more data specifically related to the conception of balance was apparent after analyzing the first few Phase Two interviews. This led to a different structure and some new questions for the later Phase Two interview.

A new question was created to start the interview. The question was designed to be a common situation in which finding averages is often learned. It also had numbers chosen in such a way as to allow the use of balance as a method for finding the arithmetic mean.

Question 1: "5 Grade Problem"

Give handout 1 to participant.

Figure **3-8**

First Grade:	90
Second Grade:	73
Third Grade:	80
Fourth Grade:	85
Fifth Grade:	97

Figure **3-8**. 5 Grade Problem.

SAY: *This sheet shows five grades of a student. Please tell me what the average grade is and how you are finding it? You may write on the handout if that would help you.*

Probes: *Why are you doing . . . ? How does that help you find the average? What are you thinking about as you do this?*

SAY: *Is there another way that you could find the average? How?* (probe as above).

This question was added to gain some contrast between the use of the algorithm for finding the arithmetic mean and the use of balance. It allowed me to return to a concrete example of finding the average in different ways.

Question 2 was the Histogram Problem. It was hoped that by placing it second more discussion would arise about connections between the actions being taken to locate the mean and the use of the algorithm earlier in the interview. In the first two interviews the task had proven to yield interesting data in that participants used several ways of thinking about the task.

Question 3 was also a new question to the interview. It was similar to Question 5, but allowed a context to be added. Because the representation was not a histogram, it allowed for different probes to be used.

Question 3: "Blood Pressure Problem"

Give the participant chart on handout 2 with blood pressure numbers.

Figure **3-9**

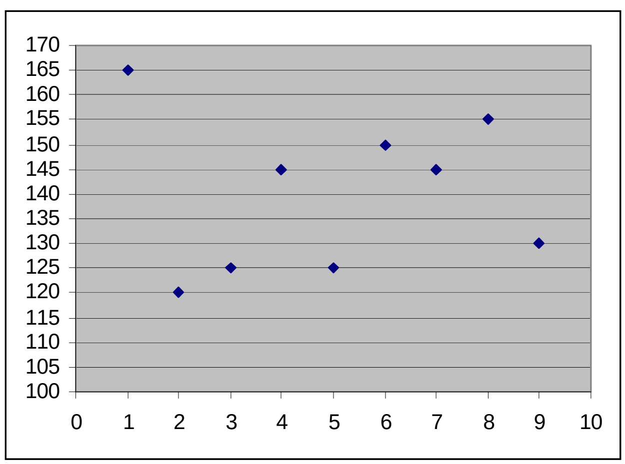

Figure **3-9**. Blood Pressure Problem.

SAY: *I have plotted nine different blood pressures (the top number) of a patient. What do you think the average blood pressure is for this patient? What are you thinking/doing to figure out what that average is?*

Probes: If participant is using algorithm, *ask if there is anything about the chart that might help them. What is it? How is it helpful?*

Cover all but two dots

ASK*: How would you find the average of just these two? Why are you doing this?* If participant equalizes deviations, reveal one more dot and ask how he/she would find the average of the three? *How is what you did different? The same?*

This question was designed to determine whether the participant could verbalize any connections between the algorithm and equalizing the deviations or balancing. The probes were meant to delve more deeply into those connections.

Question 4 was the Class Average Problem – revised and Question 5 was the Building Height problem (previously Question 4). There were no significant changes in either. However, because of the new interview schedule, more probing questions could be asked about the connection between leveling-off (the process used by most of the previous participants) and all previous methods of finding the arithmetic mean.

Question 6: "Balance Connection Problem"

Question 6 was a new question.

ASK*: When I looked up the definition of arithmetic mean in a mathematics dictionary, the definition was the algorithm: add up all the values and divide by how many data points you have. In this interview, you have been using the idea that arithmetic mean is a* *mathematical point of balance. How do you determine arithmetic mean is a mathematical point of balance from this standard algorithmic definition? Why should we expect this to be a mathematical way to represent this definition?*

PROBE: (Be sure to keep aware of whether there is a quick answer or something is being made up). *How would you explain to a 6th grader how the algorithm is related to this idea of balance?*

The final question was added to address the question raised in the previous analysis. What is that connection from algorithm to balance? It was hypothesized that *how* the participant answered as well as *what* they answered would give insight into their understanding of any connection.

3.7 Phase Three Interview Questions

The third phase of interviewing was conducted by telephone, thus tasks were not used. Rather the participants were asked questions to follow up from their previous work in the interviews. The questions (and interview schedule) are found in Appendix B. The main question was, “Could you explain how you understand that the arithmetic mean defined by the standard algorithm and the mean as a mathematical point of balance are the same thing?” The goal of the question and the follow-up probing questions was to examine whether the experts had a readily available way to connect two distinct conceptions. If the participants did not have a connection readily available, questions were asked to see if they could articulate a connection when asked to do so.

Table 1

Table 1: Summary of Interview Tasks

Tasks	Revisions	Phase of Interview
Histogram Problem		Phase One
	Added a line to divide the area in half	Phase Two
Arithmetic Mean Definition		Phases One and Two
Class Average Problem		Phase One
	Made changes to the complexity of the problem and the values used	Phase Two
Harmonic Mean Problem		Phase One
Block Problem		Phase One (early)
Building Height Problem	A revision of the Block Problem	Phase One (late)
	Changed to a bar graph representation	Phase Two
Dot Plot Problem		Phase Two (early)
Teacher Grading Problem		Phase Two (early)
5 Grade problem		Phase Two (late)
Blood Pressure Problem		Phase Two (late)
Balance Connection Problem		Phase Two
	Reworded the problem for use in a telephone interview	Phase Three

3.8 Method of Analysis

I first examined the data in a more general way. I watched the tapes and read the annotated transcriptions to get an overall "sense" of the experts' solutions. I wrote a description for each expert. This allowed me to pay attention to broad categories of conceptions contained in the literature.

Next, I chose data segments to examine more carefully. These data segments were either, "instances of competent behavior" or "unusual or unexpected responses." Among the segments identified, I set aside answers to tasks that were purely algebraic in nature because the algebra tended to mask understanding.[2]

Once I had data segments chosen, I examined them individually inferring conceptions from the experts' behaviors and explanations. These inferences were sometimes discarded or modified based on subsequent data. Others were discarded because the data were not sufficiently rich to make a compelling argument. The result of this work was further elaboration of the broad categories of conceptions found in the literature.

I next began to examine a question remaining from the initial impetus for the study. That is, how are these conceptions connected to one another? I first examined the experts' direct responses to my questions about the connections. From these responses, only C provided useful data. I used C's responses to orient a rereading of the data from the other experts. C's attention to the conception of leveling-off caused me to examine whether and how the other experts used leveling-off and whether it could contribute to understanding the conceptual connection.

[2] It also led me to change interview schedules to include the phrase "without using algebra . . . "

Chapter 4

Analysis of Data

4.1 Introduction

The current study was conducted under the premise that examining how experts understand arithmetic mean would provide a means by which I might describe goal understandings for students learning arithmetic mean. Experts from various backgrounds were interviewed. They were each asked to complete tasks that were designed to reveal how they were thinking about arithmetic mean. The following chapter is an analysis of how these experts understood the concept of arithmetic mean. I first describe two central conceptions of arithmetic mean that the experts used throughout various tasks. I then analyze connections between the two described conceptions and another way of thinking about the arithmetic mean. A discussion of those connections and the implications of those connections are then examined in Chapter 5.

4.2 Central Conceptions of Arithmetic Mean

The analysis of the data from the interviews is best organized by using two distinct conceptions of arithmetic mean that appeared to be used by the participants. The first conception is articulated in the literature but is presented in order to provide a basis for the discussion on connections between the two conceptions. This first conception is that of understanding the algorithm for arithmetic mean. The second conception is that of understanding arithmetic mean as a point of balance in the data set. In the next section I describe more fully these two conceptions. The experts viewed both conceptions as arithmetic mean but had difficulty in overtly explaining any relationship or connection between the two conceptions. Both were simply “the arithmetic mean.” I hypothesize that a critical piece of expert understanding of arithmetic mean is that the two distinct conceptions are *both* an integral part of their understanding of arithmetic mean.

In the following sections I present data from the interviews to provide a description of the two differing conceptions. Next, I present the data that led me to believe the experts did not have a readily available explanation of the relationship between the two conceptions. Lastly, I present data that led to a hypothesized connection for the two conceptions.

4.2.1 An Algorithm

The first way of conceiving of arithmetic mean is based on the algorithm. It is the result of division of the total accumulation of the data values by the number of data values. The result of the algorithm has several inherent qualities that the experts understood.

When C[3] was asked “if there was no such thing as the arithmetic mean, would you invent it and why” (Par. 110), he responded,[4] “I would invent it because it gives you an indication of a theoretical fair sharing, or equal sharing of a group of items” (Par 111). C understood that the algorithm gave a result of partitive division: fair sharing of the total accumulation.

Both B and L indicated that the arithmetic mean represented the data set best because it took into account unusual data points. B said that the arithmetic mean was

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[3] All expert participants are referred to by a letter for confidentiality purposes. In addition, the masculine form is used in all references to the participants.

[4] All transcript excerpts are edited for ease of reading. Unedited transcripts are included in Appendix C.

important because it “provided measures that in some way account for outliers” (Par 142). L stated that it “accounts for all the items involved” (Par 114). Both understood that the algorithm included a total accumulation that was, through partitive division, shared equally.

The experts conceived of the algorithm as a per-one ratio (Heinz, 2000) where the arithmetic mean represents what one data point would be if the total accumulation were equally shared. As L was explaining a difference between arithmetic mean and the median, he described this fair sharing of the total accumulation which resulted in a per-one ratio:

> Let me give you a simple example to clarify the difference [between mean and median]. Let’s say there are 10 persons in the room and each, in their pocket, have some money . . . the question is what is the average pocket money that they have . . . then we just get all the money together and divide it by 10. Okay . . . But let’s assume nine of them have less than ten dollars in their pocket and one person, they have several thousand, say ten thousand. But that ten thousand outweighs everyone . . . but if the focus is the median of the amount of the money then half of them would carry less and half would carry more. In that case, the person who has ten thousand would not outweigh the rest . . . So when you have 10 persons and you have one carry much more money than the others, and you divide it by ten that would be mostly that one person’s money, you get an answer very close to that. But in reality . . .the majority of the people don’t have anything close to that. (Pars. 116, 118, 122)

By understanding the arithmetic mean as a per-one ratio, L had an explanation for how arithmetic mean was not representative of a group with an extreme outlier. He stated that the arithmetic mean would be “mostly that one person’s money.” He understood that the total accumulation was shared equally and was supposed to represent what the accumulation would be per one person. This understanding allowed him to describe a situation where arithmetic mean was not a reasonable measure of center.

Conceiving of the algorithm as a per-one ratio; the result of partitive division, allowed the experts to anticipate how individual data points might affect the arithmetic mean. After looking at the way the data was distributed in the Histogram Problem, J said, “The right tail—any outliers will pull the mean toward it” (Par. 26). I asked him why it would pull the mean and he explained it this way:

> . . . if you have a large value in your sample, say on the right side, it can dominate all the rest of the data points. So that, in effect, you can control where you want the mean of a set of values to be by simply moving one of those values way out. The size of it in the average dominates it. (Par 28)

J viewed the arithmetic mean as two parts: a total accumulation and the amount of data points with which this would be shared. The arithmetic mean (a per-one ratio) would increase because when increasing the total accumulation the amount being shared would increase. I asked J what about the average made it so that one point could dominate. He replied, “Because the numerator is a sum” (Par 30). Although J relied on the algorithm to make his explanation, I believe he understood that the significant increase in the total accumulation would affect the arithmetic mean because it was a result of partitive division. The large accumulation of data values would be shared equally among all data points.

The experts did not appear to view the arithmetic mean as simply a number; rather they understood the implications of the algorithm. The arithmetic mean was the result of partitive division; fair sharing of the total accumulation. This resulted in a special ratio that represented what each data point would have if the total amount was shared fairly. The partitive division gave a per-one amount.

4.2.2 Mathematical Point of Balance

The second way of conceiving of arithmetic mean is analogous to a mathematical point of balance of the data values. The values are thought of in some order such as from least to greatest. These values are often related to weights and the arithmetic mean is considered to be the place in the arrangement where the set of weights would balance. In order to create this balance, the deviations from the point of balance, or arithmetic mean, must be equal on either side.

4.2.2.1 Terminology

In analyzing the expert's work with balance, the word "weight" was used in several different ways. In this section, I will define several words that more accurately reflect how the word "weight" was used throughout the interviews. These terms are then used in describing the use of "weight" within the analysis of the interview data.

Mass: In physics, the term mass is most accurately defined as a quantity of matter in a body regardless of its volume or any forces acting upon it (Columbia Encyclopedia, 2004). However, in Standard English usage, the term weight is often used interchangeably with mass because it is standard to measure mass by finding the weight of an object (Borowski & Borwein, 1991, p. 363). I did not interpret any use of the word "weight" as mass in the study; however, it is included to offer a possible interpretation of the word.

Weight: A measure of the force of gravity on an object. Weight is often used to measure mass. In statistics, weight is thought of as the number of cases of a particular data value.

In the analysis I will distinguish between physical weight, an attribute of the balance analogy and statistical weight, an attribute of the data themselves.

“Physical weight”: refers to the standard measure of gravity on an object.

“Statistical weight”: refers to the number of cases of a particular data value.

Deviation: According to the American Heritage Dictionary, deviation is “the difference, especially the absolute difference, between one number in a set and the mean of the set” (2000). Deviation usually refers to any difference between two values, but in statistics it specifically refers to the difference between a data value and the mean of a set. This statistical definition of deviation was used by most of the experts when referring to the distance (sometimes absolute) of data values from the arithmetic mean.

Torque: “The moment of force; the measure of a force’s tendency to produce torsion and rotation about an axis, equal to the vector product of the radius vector from the axis of rotation to the point of application of the force and the force vector” (J. Pickett et al., 2000). Torque may also be considered as the force that is applied perpendicularly to a beam. It is calculated using the multiplicative relationship of force (or weight) and distance from a point (deviation). There were several instances in the interviews when the experts referred to “weight” yet appeared to be describing torque; such occurrences in the analysis of the interview data are labeled as torque.

4.2.2.2 Balance is Equalized Deviations

The experts in the study used the word “balance” to describe the work that they were doing with the arithmetic mean. B stated, “So, if that’s true, then this part would *balance* out this part" (Par. 21). K described the arithmetic mean as "almost like the *balance* point of a teeter totter" (Par. 50). C used his conception of balance to try to find a placement for the arithmetic mean in the Histogram Problem. He said,

> Well, I'm thinking in terms [places pen vertically on graph on the third bar] of a *balance* . . . where do I need to balance this, where would this balance . . . Where could I put a fulcrum that would level these out? (Pars. 47, 49)

In defining arithmetic mean, J drew a picture (Figure **4-1**) of arithmetic mean as a point of balance. As he drew, he said,

> Suppose you have three data points [draws line and places three hash marks on the line]; one there, one there, and one there. And you put a pound of clay there and pound of clay there and a pound there [pointing to hash marks]. And you find a point [draws a triangle under the line] at which it *balances*; that will be the mean. (Par. 72)
>
> Figure **4-1**

Figure **4-1**. J's Drawing of Balance.

All of the participants used the term balance when discussing the arithmetic mean. The activity that was most often related to the use of this term was that of equalizing the deviations from a proposed arithmetic mean. When the participants were attempting to find that point of balance, they would propose an arithmetic mean, and then check to see if the deviations on either side of the arithmetic mean were either equivalent or cancelled out one another. If the deviations were equal or cancelled out, then the proposed point was correct. If the deviations were not equal (or did not all cancel), then the proposed mean was moved to cause the deviations to equalize. These activities relate to the concept of torque in that they both create equivalent torque on either side of the arithmetic mean. The deviation and statistical weight combination on either side of the arithmetic mean (which acts as if it is a fulcrum) are equivalent.

As an example, L equalized the deviations in the Building Height Problem. He was given a bar graph that depicted building heights and was asked to find the average height of the buildings (Figure **4-2**).

Figure **4-2**

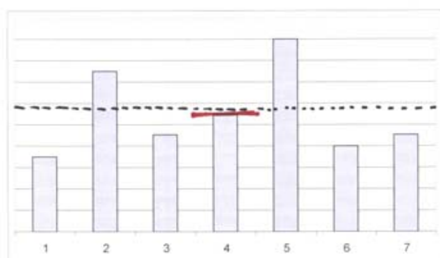

Figure **4-2**. L's Average Building Heights.

First, L came up with an estimate for the average building height. He then began to check to see how accurate his estimate was. Because he was not clearly stating what he was doing, I clarified his work and he agreed with my statement. He stated,

> L: So, anyway, by adding those shorter than the average, this is not the – the off-estimate [pointing to dashed line, Figure **4-2**]. I counted five and this [moves finger below the dashed line] this is six or seven. So, it should be a little bit lower than that [points to just below the dashed line].
>
> I: Okay, so what you were doing was – you were – correct me if I'm wrong. But, I think I was understanding that you were taking how far it was from the top of each bar to your estimated average. And you're trying to make what's between here and here [points to top of bar 6 and proposed mean] the same as between here and here [points to proposed mean and top of bar 5]
>
> L: Exactly. (Pars.146-148)

Assuming that L's comment that this was "exactly" what he was doing, L was focusing on the deviations from the proposed mean with the goal of getting the sum of those above equivalent to the sum of those below. L refined his description after he refined the positioning of his proposed mean.

> So if I add the one, two, three, uh three probably point four [adding spaces above the new proposed mean on bar 5, Figure **4-2**] and this is probably one and two [pointing to spaces on bar 2 above the proposed mean] so 6.4. So, the two, the tall buildings we have a combined, 6.4 units tall. . . . And the four shorter building, where this um might be like 1.6 [points to space above bar 6 to new proposed mean], this would be very close to 1 [points to bar 7] so 2.6 and this is 3.6 or 3.7 [points above bar 3] and 5 and 6.7 [pointing above bar 1]. So, the two are very close. (par. 150)

L was adding up the deviations from his proposed mean and comparing those deviations above the proposed mean to those deviations below the proposed mean with the goal of making those deviations the same. To L, this activity gave him the average building height.

As C worked on the Histogram Problem (Figure **4-3**), he illustrated a second way of obtaining equalized deviations.

Figure 4-3

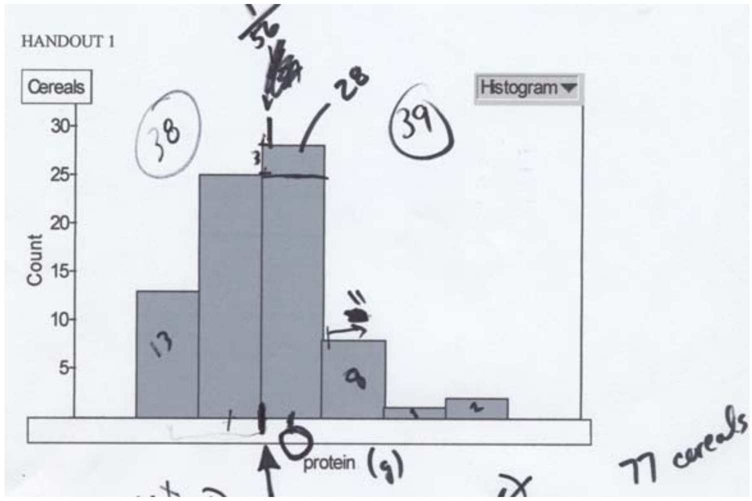

Figure **4-3**. C’s Histogram Problem.

As stated earlier, C was searching for where a point of balance for the data set might be. To find the point of balance, C represented the deviations from his proposed mean numerically and then “cancelled out” positive and negative deviations. He said:

> So we have a distance here [draws horizontal line lightly from midpoint of 1st bin to the dividing line, Figure 4-3] of one and one half, a distance here of half [points to midpoint of 2nd bin]. We’ll call those negative [draws parentheses around values and adds negative signs, Figure 4-4]. And then we have 28 times a half, 8 times one and a half and 2 times two and a half. And that sum [draws line under list in Figure 4-4] should be zero. (Par. 56)

Figure **4-4**

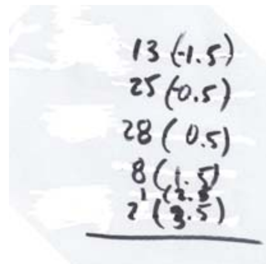

Figure **4-4**. C’s Numerical Deviations.

C continued his work and cancelled out negative deviations with positive deviations:

> So, I’ve got eight here [crosses out 8(1.5) and 13(-1.5) and writes 5(-1.5) to the left], that leaves me with a net of five, negative. And then, let’s see, I’m just going to add these [has pen over 1(2.5) and 2(3.5)] up to see. We’ll let’s see, these [crosses out 3(.5) and changes 5(-1.5) to 4(-1.5)], knock this down to a four, um, and then seven, nine and a half [pen is over 1(2.5) and 2(3.5), see Figure **4-5**]. (Par. 74)

Figure **4-5**

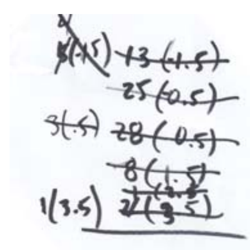

Figure **4-5**. C’s Cancellation Process.

C’s goal was to have the weights he had numerically described cancel out with one another. Although this did not occur with his proposed mean, C’s calculations illustrate how he would find a point of balance.

Both methods—summing deviations over and under and comparing or cancelling negative deviations with positive deviations—resulted in having the deviations from the proposed mean equalized. A hallmark of all of the experts' understanding of balance was that balance was not some notion based just on physical experience, but rather was a mathematically precise notion of equivalent deviations. The activity of equalizing deviations was how the expert found the mathematical point of balance.[5] Because a mathematical point of balance was the point at which the deviations on either side of that point were equivalent the experts understood the statistical situation as analogous to a situation of torque in which static equilibrium was created.

4.3 Two Distinct Conceptions

What is the connection between the conceptions of arithmetic mean as balance and understanding the algorithm for the arithmetic mean? Several of the probing questions in the interview schedule were designed to have the expert participant explain a connection between the two conceptions. I anticipated asking questions such as those used in the Building Height problem, "How does this idea of mean as a balance point relate to the definition of mean that you gave to me earlier?"

After gathering little useful data in the earlier interviews, I attempted to more carefully phrase the questions in order to get some insight into any connections between these two understandings in the later interviews. The final questions of the later interviews were "How do you determine arithmetic mean is a mathematical point of

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[5] Although physical experience appeared to allow the experts to estimate the point of balance with a great deal of accuracy, it was simply an estimate to them until they found the point at which the deviations were equal.

balance from this standard algorithmic definition? Why should we expect this to be a mathematical way to represent this definition?" During the later stages of the study, I focused the interview in order to better understand this connection.

There were several places in K's interview where I asked him to connect his understanding of the algorithm with his understanding of balance. In the 5 Grade Problem (See Appendix B.2), he was asked how he would find the average of the five grades. His first method was "I would add up all the numbers and divide by the . . . total number of numbers" (Par. 10). When asked if there was another way, he found a middle value and then said, "I would balance the distance below with the distance above" (Par. 16). K clearly had both conceptions as a way to find the arithmetic mean. That is, there appeared to be a procedure-concept connection that was unique to each procedure. I next asked him, "Can you describe how that is similar to adding everything up and dividing by 5? . . . Is there any way you can describe how what you just did, where you got the points below and above the same as, or is similar to . . . " (Par 25). K's reply was, "I would say, adding them up and dividing by 5 is just a short cut to doing it the weighting way" (Par 26). At this point, the reply implied that there was not a way to articulate a connection that illustrated how both procedures were essentially the same thing

The initial response was that the two were the same thing, although one was a shortcut for the other. I did not follow up at this point in the interview because I knew I would be asking for more clarification later in the interview. K next solved the Histogram Problem, the Blood Pressure Problem and the Class Average Problem. In each case he used a variant of equalizing the deviations in order to create balance. For example, in the Blood Pressure Problem K said:

> I could measure the distance. So 25, [writes this on one from dot 1 to drawn line, Figure **4-6**] and 5 [dot 4] and add up these distances [runs fingers along dots above the line] and measure these [runs finger along dots below the line] and add them up. And if they are the same [points to above the line and then below the line], then I'm good to go. (Par. 92)

Figure **4-6**

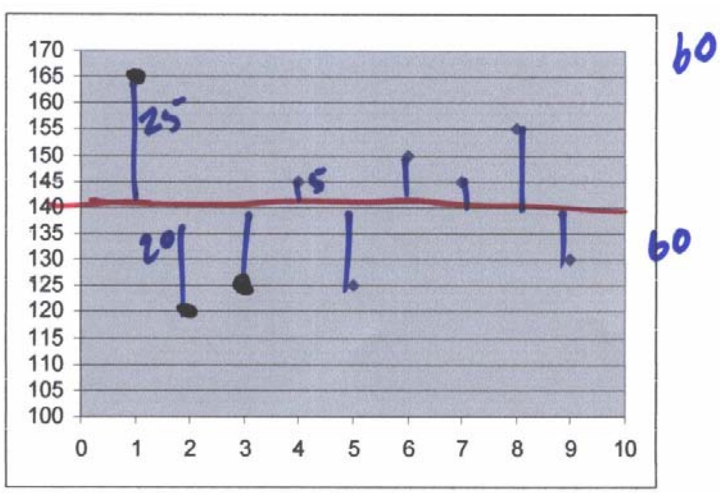

Figure **4-6**. K's Blood Pressure Problem.

After completing the three tasks using his conception of arithmetic mean as a point of balance, I asked K:

> When I first asked you to find the arithmetic mean you said, "I'm going to add everything up and I'm going to divide by how many there are." . . . However, in this interview, for every single one of these problems that we've worked on you have referred to it as a mathematical point of balance, in a sense. In this one [points to histogram] you said it's a place where, like a teeter totter it balances . . . my question is, how do you determine that the arithmetic mean is a mathematical point of balance from this standard algorithm that we're so used to using? (Pars. 169, 171, 177)

K used the 5 Grade Problem in order to explain the connection. First, he used the algorithm, "so the sum is 425. Divide by 5, OK 85" (Par. 182). Next, he tried to pull together a connection,

> OK, so this is the mean [writes $\bar{x}$, Figure **4-7**] because we have calculated the mean. We know this is the center point [points to 85] but how we can use that is because we know that if we say the distance between this [draws line from 85 to 97 and labels as 12] is 12. And the distance between . . . plus 12 [add "+"]. And the distance to this [draws line up to the 90 and labels +5] is plus 5. We've got [writes an 85 below the grade list, draws a horizontal line above and then writes 17 to the right] 17 on this side [adds a carat mark just below the 85, no comment with this]. And we know that we've got [points between 80 and 85] negative 5 here [writes -5 between two numbers] and negative 12 [points between 85 and 73, writes -12] so we've got negative 17 [writes on the left hand side of the horizontal line]. So that's why the weight is the same. (Par. 184)

Figure **4-7**

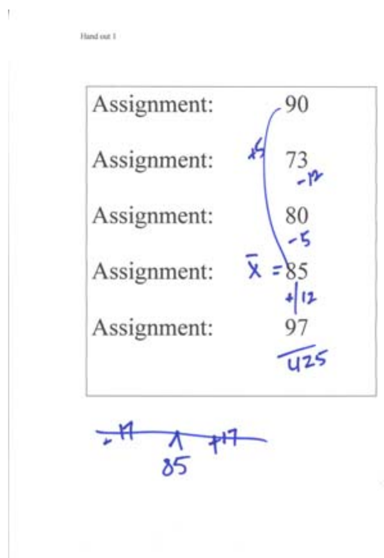

Figure **4-7**. K's Explanation of a Connection.

K's work simply proved that the two strategies each yielded the same answer. I asked K, "Have you ever been asked to figure out why it is a mathematical point of balance?" (Par. 185) and his response was "No, I don't think so" (Par. 188).

I made one more attempt to probe into connections, going off the written interview schedule. I tried to explain more clearly to K what I was trying to understand by reiterating the problem using the 5 Grade Problem K had used earlier. I said:

> I still don't see how when I take 73 [begins to write column addition] 80, 85, 90 and 97 and I add those up and get whatever it was . . . 425. Then I take that 425 and divide it by 5 to get my 85 [writes division problem, Figure **4-8**]. I'm still not sure how this algorithm . . . can reduce to this [points to drawing of balance above list] because when I reduce to this [begins to add arrows] I'm looking at . . . you've been looking at this is . . . plus 5. And this is plus 12 [drawing arrows and labeling] and . . . this is minus 5 and minus 12. And you're looking at that. And I'm not really sure how this [puts hand over lower half calculations] is related to that [puts hands over the number line].Truly, I don't know. (Pars. 253, 255, 257)

Figure **4-8**

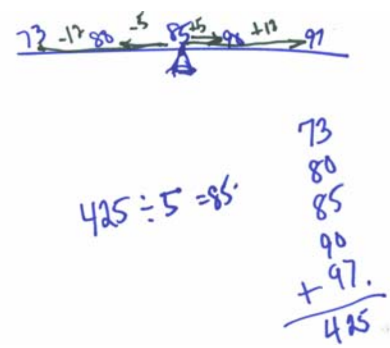

Figure **4-8**. Interviewer's Connection Drawing.

K's response was "It's magic" (Par. 258), but then tried to answer again saying:

> Let's see. If I thought of these as weights instead of scores, like I have a 73 pound student and an 80 pound student . . . [talking very slowly and drawn out] then these, in order for this [points to 85] to be the middle point I have to come up with two students [points to left side of number line] to weigh exactly the same as [now points to right side of number line] these two other students. (Par. 262)

This still did not seem satisfactory as an answer, so I continued to ask saying, "What I'm trying to understand is how we come up with this idea of balance because we use it all the time . . . " (Par. 263). K's conclusion to this was "Well, this is like the shortcut [runs finger down calculations] to, and once you get the number, then you look at [runs finger along number line]" (Par. 264).

The telephone interviews were conducted because K was the only expert to this point in the study who had been questioned about a connection between these two distinct conceptions of arithmetic mean. Each expert who had previously been interviewed was telephoned and asked a series of questions about any possible connections between the two conceptions (see Appendix B.3 for the full telephone interview schedule). These experts were asked, "Could you explain how you understand that the arithmetic mean defined by the standard algorithm and the mean as a mathematical point of balance are the same thing?"

In his interview, J responded:

> It is almost as if those two ideas come from very different sources. The point of balance idea comes from the idea of a centroid, I believe. Like a teeter totter . . . I'd have to think about that. Because one of them is a mechanical/physical idea and the other is algorithmic. And I don't think there is any obvious way to connect between the two. (Pars. 16, 18, 20)

Throughout the telephone interview with J, no connection was articulated and J was quite clear that he believed that, although the question was interesting, he had no way to think of the two distinct ideas as connected.

Both L and B were able to express an algebraic relationship, but when asked to describe the connection further, they were unable to do so. L described the connection in this way:

> Beginning with the equations – you have the summation of all the data points divided by the total number of data points so you are looking at the true average . . . Now if you multiply both sides by the total number, the number of data points . . . .And then you move the left hand side of the equation which represents the summation of those *n* data points [to] the

> right side. Then you have the, you would take the *n* different terms. The difference between the average and the individual data points . . . .and then the summation must be zero . . . .That means the point of balance must be the true average such that the summation of all the individual points [deviations] would be zero. (Pars 36, 38, 40, 42, 44)

L was stating symbolically:

$$\frac{\sum x_i}{n} = \bar{x} \quad \textit{multiply by n}$$
$$\sum x_i = n\bar{x} \quad \textit{or equivalently}$$
$$\sum x_i = \sum \bar{x} \quad \textit{subtract each } x_i$$
$$0 = \sum (\bar{x} - x_i)$$

His conclusion was that the algebraic relationship showed that the two conceptions were equivalent. Yet he could not describe any connection beyond this algebraic manipulation.

B initially responded that he would connect the two conceptions symbolically or with an algebraic representation. After discussing this he was asked, “Is there anything in the algorithm that suggests to you that that should be a mathematical point of balance?” (Par 31). B’s response was:

> Well, I think the way I probably think about it is that the mean is a distance. Well, not a distance but that the distances between the mean and each point balances out. In the actual add them up and divide by the number of numbers you have – as it is I think you’d have to do some transformation. So is there . . . “anything in there?” Uh, I think I think about average as this notion of – well I can think of it as leveling off or balancing or . . . [fades out and does not continue]. (Par. 32)

The notion of leveling-off appeared to be connected to B’s conception of both balance and also, possibly, to the conception related to the algorithm. However, B did not give a description of a connection other than the algebraic connection.

C was the one expert who described some type of relationship between the algorithm for arithmetic mean and arithmetic mean as a mathematical point of balance. It appeared that he was connecting leveling-off with each of the two differing conceptions of arithmetic mean. The algorithm collected all of the accumulation and equally shared that among each data point. Leveling-off was a way to think about arithmetic mean that accomplished the same thing as the algorithm. C said in the telephone interview, “We are adding up the total quantity . . . and dividing it equally among five different spots. So you can think of that maybe as bars of a bar graph and you are getting them all to the same level” (Par. 28).

C also connected leveling-off to the idea of balance. He continued:

> C: If I put five dots on here [a drawn balance beam he had described], randomly—two on one side: two to the left of the fulcrum– three to the right of the fulcrum. The balancing – I mean the idea of leveling is I guess, moving them all to the – well, let me take away the fulcrum because that makes it harder if you’re trying to move it to a preconceived spot. So have a line – balance beam, no fulcrum yet. And, I’m moving these dots that are randomly spaced along the line to a point. And that point would be where the fulcrum should be placed.
>
> I: OK. How are you choosing where to move them?
>
> C: Well, it’s going to have to be in terms of what I’m – the distance I’m moving one dot on the right would have to be equal to the total distance I move dots on the left. (Pars 32-34)

C called this leveling-off because it was again taking a prescribed amount from one side and “moving” it to the other side. In this instance it was the distances (or deviations) that were the subject of the leveling-off.

To describe a connection between these two separate ideas, C initially used a visual representation of the bar graph and the dot plot. However, as he explained further, he believed that a numerical representation made the connection more apparent. He said:

> So if you read off like a list of grades – lets say they were 88, 86, 82, 78, 76 . . . .And I was going to calculate an average. Then I could kind of ball park some kind of midpoint as my target. So let's say, 82! [there was a pause before picking 82 and a voice of surprise] Which actually looks to be exactly what the average is. Then I would say well 88 minus 82 that's minus 6. So, I've got to add 6 to another number. So I would have that my 88 is now an 82, my 76 would now be an 82, I would have an 82 and my other two numbers are 86 and 78. So to make 86 [into] 82 I've got to go down four. And then go up for from 78 . . . .So, in that sense, I see those two processes as identical, just a different representation of the same process. In terms of moving the dots to that same point and calculating an average the way that I did . . . .Now in terms of the connection to the standard algorithm, it might be easier to connect my numerical leveling to the standard algorithm rather than the visual . . . So . . . we're turning the process into —calculating the arithmetic mean into a series of additions and subtractions . . . (Pars. 42, 44, 46, 50)

C tried to describe how subtracting and adding (which he also referred to as leveling-off) was one way to find the point of balance, but also was similar to the algorithm of adding everything and dividing. The following conversation ensued:

> C: I don't have a clear way to articulate that connection.
>
> I: Is there anything in the algorithm that would suggest balance to you?
>
> C: Yeah – the division
>
> I: OK, why would division suggest balance?
>
> C: Because you are partitioning a number into – a total into x number of equal parts. In my case, five equal parts.
>
> C: Why would division suggest balance?
>
> P: Because you are *partitionin*g a number into – a total into x number of equal parts . . . . I think that if you go back to my description of moving the dots to a balance point. It's kind of the – it doesn't immediately make me think of balance . . . .I think leveling is more of the connecting point for me. (Pars. 52-60)

C had to work to make an overt connection. However, he saw leveling-off as a connection between the two conceptions.

The data show that each expert could conceive of the algorithm for arithmetic mean and arithmetic mean as a mathematical point of balance. With the exception of C, I was unable to get the experts to explain a connection between the two conceptions in the context of the interviews. C connected his understanding through the use of leveling-off.

4.4 Use of Leveling-off as a Connecting Strategy

4.4.1 Leveling-off Described

Leveling-off was used to find the arithmetic mean in several different representations given in the interview tasks. Leveling-off can be used with a bar graph representation of *individual* data points. Each bar's height represents the value of one data point. The bar graph of individual points is not a typical representation for data points in a data set. When using leveling-off, the total value of all the bars is never changed. Instead, taller bars have pieces "taken off" and shorter bars have these pieces added to their bar. The activity of moving pieces is continued until the bars are all the same height (see Figure **4-9** for a visual description). The Building Height Problem provided a bar-graph representation of this type. The use of this representation in the interview allowed me to gain insight into expert reasoning about leveling-off that may not have been apparent in more standard representations.

Figure 4-9

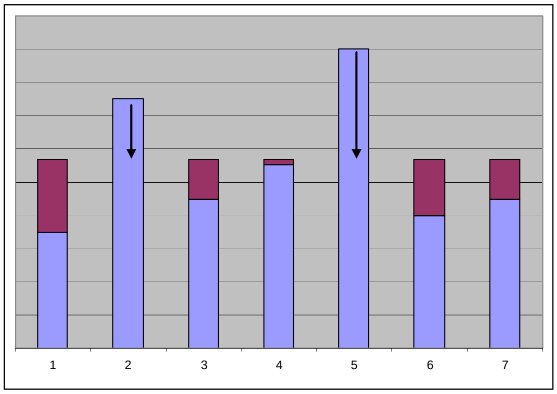

Figure **4-9**. Leveling-off Illustrated.

Sometimes the experts thought of leveling-off as an activity to find the arithmetic mean as a mathematical point of balance. At other times the experts thought of the leveling-off activity as the same activity as redistributing the total accumulation so that it was shared equally.

4.4.2 Leveling-off to Find the Mathematical Point of Balance

The experts considered arithmetic mean as the balance point of a data set as equivalent to the point at which the deviations from that point were equal. In some instances their actions implied that they were leveling-off although they did not necessarily use this terminology. The experts viewed the activities they were doing as balancing the data set.

In the Building Height Problem, B used leveling-off in order to find a point of balance. B started by stating that he was balancing the set: “I could do some balancing” (Par. 197). Next, B described how he would accomplish this saying, “I want a line so that the part that’s cut off above, I could transfer to whatever places below. In other words, I want to level it" (Par. 197). When B was asked if he could describe the "balancing" within the representation he said,

> Yeah . . . Because what you'd be – All right, I have to think about this a little bit [Holds bar graph ¼ turn making vertical axis now horizontal] so the m minus the x, I'm doing, you're saying the absolute values here . . . so we're just talking about the distance here [draws a line on bar 2 that spans from the horizontal line to the top of the bar] needing to equal the distance from here, to , you know, in this case, here [draws from horizontal line down to top of bar 1,Figure **4-10**] (Pars. 224, 226)

Figure **4-10**

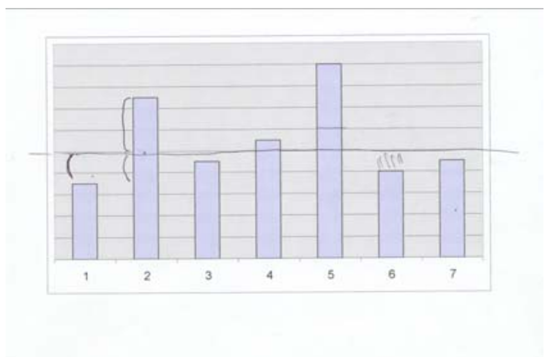

Figure **4-10**. B's Balance of Buildings.

B described balancing as the act of making the deviations equal. He suggested that when the deviations (represented by amounts over and under in the bar graph) were moved around it was the same as having the deviations on one side equivalent to the deviations on the other side. Thus, by the action of leveling-off, B had found the mathematical point of balance. Both K and L also used leveling-off to find a more accurate arithmetic mean in the Building Height Problem. They used an explanation similar to B's when they explained how they were "balancing" in the Building Height Problem.

In a weighted data situation such as the Histogram Problem (see Figure **4-11**), leveling-off was not an obvious choice for finding the arithmetic mean. The height of each bin of the histogram represented the number of cereal boxes at a particular protein level. The experts were asked to find the average protein level not the average number of cereal boxes. Thus, leveling-off would not equally share the grams of protein. The experts understood this and worked to balance the distribution of protein values rather than leveling-off the heights.

Figure **4-11**

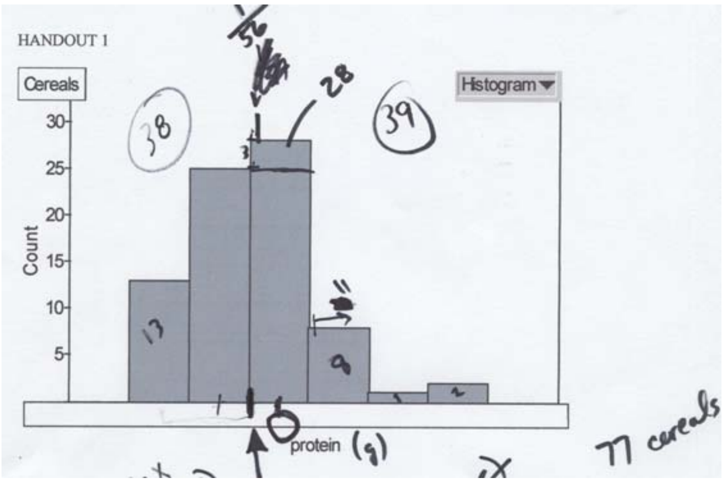

Figure **4-11**. C's Histogram Problem.

Despite the difficulties in using this representation, C's work with the problem highlighted a connection between his conception of arithmetic mean as balance and leveling-off as a strategy to find the arithmetic mean. To best understand C's work, I first present the context which led to his use of leveling-off. After describing his work leading up to leveling-off, I highlight how C connected leveling-off to his understanding of arithmetic mean as balance.

C had represented the deviations from his estimated mean with positive and negative numerical values. Because the data was statistically weighted (there were several values at each protein level), C then multiplied by the height of the bin in order to find the total deviation from the estimated mean for the bin (see Figure **4-11**). His final calculation is shown in Figure **4-12** . C next began the process of cancelling out the positive deviations with the negative deviations in order to find if his proposed arithmetic mean was the point of balance.

Figure **4-12**

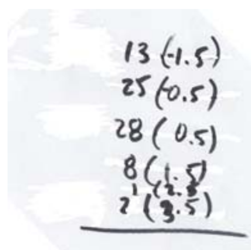

Figure **4-12**. C's Histogram Deviation Computations.

When he completed his work he had a result of a positive 3.5 deviation. He said, "So, we have, this was our proposed mean [points to line on histogram, Figure **4-11**]. So the sum of the deviations from that point is 3.5. If we divide that by 77 and add it to that, that's where the mean is" (Par. 83).

Up to this point C's work and description appeared to be a balance strategy. He was trying to find a point of balance, so he estimated and checked to see if his deviations were equal. When he discovered they were not, he had a method for fixing his estimated mean. He would divide the resulting deviation by the total number of cereal boxes and add that to his estimated mean. I asked him "why do I have to divide it by 77?" (Par. 84) At first C replied,

> this is the way I used to average grades in my head – select a value that I thought was close enough and then do the sum of the deviations. And that remainder, you have to divide by the total number because that has to be shared among all the values. (Par. 85)

It is not unexpected that an expert would have a method for finding a value without a readily available explanation since he would have most likely compacted his understandings.

As C unpacked his method, he revealed a way that he connected his balancing activity to the activity of leveling off. C's initial focus was on the deviations since that was the focus of his numerical activity. He said, "Basically each of these deviations is attached to a value, so there is no sharing involved, it is total value. But the sum of all of the . . . " (Par. 85) and then C paused. During this pause, it is likely that he was thinking of how to explain the "sharing" of the left over deviation. He understood that the values were being accumulated, but he also was thinking of how his deviations were to be equalized (summed to zero). C moved to the activity of leveling-off to explain his work. He said, "So, if I was thinking of leveling here [lays pen horizontally across the histogram] well, they are drawn the wrong way for leveling – I'm basically leveling" (Par. 87). C re-represented the data as non-weighted data allowing him to more clearly explain what he was numerically doing when he shared the leftover deviation. He said,

> So, let's put my protein values over here [draws a vertical line and labels it protein, Figure **4-13**] So I'm basically making, what do I have to do to each value [runs pen along deviations calculations] to get it to this value [points to proposed mean on the histogram]. This is my proposed mean [from the vertical line labeled protein he draws a dotted horizontal line across]. So, I've got some values that don't reach it, I've got others that go

> above it [draws a bar that is below the dotted line and a bar that goes above] . . . (Par. 87)
>
> Figure 4-13

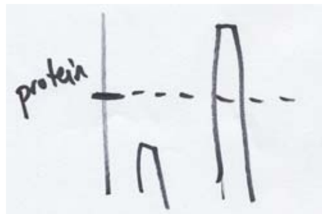

Figure **4-13**. C’s Re-representation of the Histogram Problem.

C understood the statistically weighted data as a particular number of non-weighted data. He stated that he had been doing his work in shorthand, but what he was doing was leveling-off. Thus, C viewed his numerical cancelling (a strategy for confirming the point of balance) as equivalent to leveling-off. He created a set of all bars of the same height without using up all of the total accumulation. One bar had the extra accumulation depicted on it and was higher than his proposed mean. Having an extra 3.5 deviation meant that balance had not yet been achieved because the bars were not all the same height. He then took the remaining total accumulation that was above the proposed mean and divided it up equally among the 77 bars to achieve balance.

In his telephone interview, C once again connected leveling-off to balance. At first he said, “I don’t know if I can make the same connection to the leveling-off with the point of balance. But I’m drawing some pictures here. . .” (Par 22). However, as he continued to talk he used a dot-plot representation in order to make a connection.

> C: So I have a line – balance beam, no fulcrum yet. I’m moving these dots that are randomly spaced along the line to a point. And that point would be where the fulcrum should be placed.

I: How are you choosing where to move them?

C: Well, it is going to have to be in terms of – the distance I'm moving one dot on the right would have to be equal to the total distance that I move dots on the left . . . and so I would keep moving. . . Let's say these are all equally weighted dots. If I move a dot one unit on the left, then I've go to move a dot one unit on the right. And so I could continue to move dots on each side until I got all of them to the same point – all stacked up at the same point.

I: And so that's like leveling in what way?

C: That's like leveling in the sense of when I think of calculating an average, an arithmetic mean; I will take values away from one number. Take an amount from a number and add it to another number. And I'm looking for that point that the two – or that all the numbers would be the same value (Pars 32-38).

Each distance C moved a dot in from the right side toward the middle had to be matched with a move he made with a dot on the left side. This meant the total distance on either side had to remain the same. C called this leveling-off because he imagined that he was moving a positive deviation and placing it in a negative deviation's position. This was continued until all of the dots were in the same place. That place represented the arithmetic mean.

4.4.3 Leveling-off Connected to the Algorithm

C also connected his understanding of arithmetic mean as an algorithm as he worked with the Histogram Problem. As described in Section 4.4.2, C had connected leveling-off with finding the point of balance by numerically cancelling in this task. After he had described the connection to finding a point of balance, he still had positive accumulated deviations so he did not yet have the correct arithmetic mean (or point of balance). C tried to explain why he took the extra deviation and divided it evenly among the 77 boxes of cereal. He drew one more representation to help explain his reasoning.

> So, I'm doing it in short hand a little bit because I'm dealing with weights rather than dealing with each one individually and cancelling it. So, this is basically a leveling process for some proposed mean but when I'm left, I've got one, I've got [redraws the previous figure with three bars that come up to the dotted line and one bar that is over the dotted line, Figure **4-14**] – after I've kind of done this cancelling out process, then basically I've gotten all my values level to this, except I've got one value now that is 3.5 above [labels section of bar above the dotted line as 3.5]. So, I've got to share this [points to bar above the dotted line and then takes hand and stretches along dotted line] among all 77 values to get the actual mean. Because this [points to dotted line] is not the mean. (Par. 87)

Figure **4-14**

Figure **4-14**. C's Second Representation of the Histogram Problem.

The re-represented data allowed C to describe the connection between what he had been doing with cancelling out of the deviations (balance) and his method of dividing the remaining deviation by the total number which seemed to relate more closely to the result of partitive division. As he explained, the cancelling of deviations was the same activity as making all of the bars the same height. That is, finding a height that represented the result of equally sharing the total accumulation. However, the extra deviation meant that one represented value was still as if it had too much of the shared accumulation. In order to get all of the accumulation as if they were the same and consequently also have the deviations equal, C had to divide the remaining deviation by the total number of data points and then add this small amount to his estimated mean.

This new mean represented the total accumulation properly shared among all of the data points; that is, partitive division of the total accumulation.

In the telephone interview, C again connected leveling-off to totaling up all of the data values and dividing it equally between the [five] spots. He stated:

> I'm thinking visually in terms of when I was talking about the algorithm I was—I made dashes, kind of like hangman space . . . We are adding up the total quantity. I don't have a better term for that. And dividing it equally among five different spots. So you can think of that maybe as bars of a bar graph and you are getting them all to the same level. (Pars. 26,28)

Because the total amount was being equally shared in the division it was equivalent to making all of the bars the same level. C thought of the level bars as representing the result of the partitive division.

When B had completed his work to find the arithmetic mean in the Building Height Problem, he was asked how his work of moving deviation pieces was similar to other descriptions he had written of arithmetic mean.[6] He stated:

> I don't think I've accounted for that in these [referring to written definitions]. Because what you're doing is you're taking some part of that distance and moving it someplace else [points to bar graph] (Par 213).

B viewed his activity in the Building Height Problem as "moving" distances (deviations) from one place to the other. He had previously said he wanted to "level it" (Par.197). I asked him how his work of leveling might be related to other descriptions of arithmetic mean, suggesting he make something up since he had not accounted for it earlier. As he thought, B placed his finger on an equation (see Figure **4-15**) that he had written to describe the algorithm of arithmetic mean and said, "I think it's closest to this" (Par 216).

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[6] B's descriptions of arithmetic mean were all algorithmic in nature rather than verbal descriptions.

Figure **4-15**

$$M_{x_i} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Figure **4-15**. B’s Description of the Arithmetic Mean Algorithm.

B worked between his written equations and the bar-graph depiction of the Building Height Problem and created a new equation to illustrate how he believed his work in leveling-off was related to the algorithm of arithmetic mean. He first wrote the equation shown in Figure **4-16** to illustrate how he was thinking of leveling-off.

Figure **4-16**

$$n M_{x_i} = \sum_{i=1}^{n} x_i$$
$$x_1 - a + x_2 + a + x_3 - b$$
$$+ x_3 + b \ldots$$

Figure **4-16**. B’s Formula for Equalizing Deviations.

Next, B explained how this equation was represented in the work he had done in the Building Height Problem. He said:

> I've got my average height which is this [points to the horizontal line drawn on the bar graph for the proposed mean, shown Figure **4-17**] times *n* which is my number of buildings [runs pen along each bar]. [This] is the same as the sum of [runs pen up each bar in bar graph] each of these heights. It doesn't tell me exactly that I'm taking this piece [spans finger along top of bar 5 that is above the proposed mean] and moving it here [now spans finger from the top of bar 6 up to the proposed mean] but that's what that would mean is within here [points to summation notation, Figure **4-16**] what I'm doing is I'm taking something away from x_1 and adding that to say an x_2 . . . and so on, so I keep the x_1 plus x_2 through n. (Par. 218)

Figure **4-17**

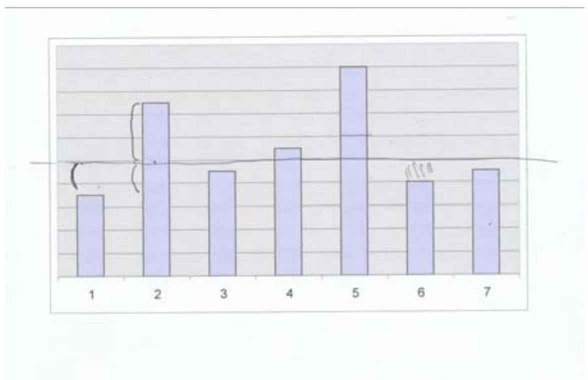

Figure **4-17**. B's Building Height Problem.

B described algebraically how the sum or total accumulation of the data set did not change, but rather the data values were adjusted by reducing some values and increasing other values by an equivalent amount.

To B, the key idea in relating leveling-off to descriptions of the arithmetic mean appeared to be that the total accumulation of data values remained the same in each approach. When he leveled-off, the data values that were adjusted by reducing exactly matched those data values that were adjusted by increasing. B pointed to a leveled-off bar and stated that the height of that bar times *n*, the number or buildings, would give the same amount as totaling the heights of each individual bar. This was a way to describe a per-one amount represented by leveling-off. The algorithm also kept the total accumulation of the data values the same. The equation he presented in Figure **4-16** had the arithmetic mean multiplied by the number of data points (*n*) on the left side. The right side of the equation was developed to illustrate how the "pieces" were added or subtracted from a bar and then subtracted or added to another bar. B's conclusion was that this total accumulation did not change. The total accumulation remained the same, but how that accumulation was shared was changed. Thus, leveling-off was equivalent to partitive division.

4.4.4 Leveling-off as a Connection Between the Two Conceptions

The data analysis led me to believe that not all the experts had an overt[7] way to connect the two differing conceptions of arithmetic mean as I have articulated them. K jokingly termed it "magic" when asked how the algorithm was related to the idea of mean as a point of balance. After J heard that I was trying to figure out what a connection might be he said, "That's an interesting question, I think, because it certainly isn't obvious to me" (Telephone interview, Par. 37).

However, C's work suggested that leveling-off was a possible point of connection; a procedure that could be related to each conception. In the telephone interview, C was asked to describe the connection between the two separate ideas of understanding the algorithm for arithmetic mean and arithmetic mean as a mathematical point of balance. To answer, C changed the representations he had been using but

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[7] In the analysis of both B and C's task-based interview, I was able to describe possible connections. The experts themselves did not state any connection either because they were not asked (C) or they did not see one at that time (B).

continued to use the leveling-off strategy as a point of connection. C believed that the numerical representation helped to make the connection more apparent. As shown in Section 4.3, C set up a numerical scenario to describe how leveling-off was like his understanding of balance and his understanding of the algorithm.

> So if you read off like a list of grades – let's say they were 88, 86, 82, 78, 76. . . and I was going to calculate an average. Then I could kind of ball park some kind of midpoint as my target. So let's say 82. Which actually looks to be exactly what the average is. Then I would say well 88 minus 82 that's minus 6. So, I've go to add 6 to another number. So, I would have that my 88 is now an 82, my 76 would no be an 82, I would have an 82 and my other two numbers are 86 and 78. So to make 86 82 I've got to go down four. And then go up four from 78. . . So, I see those two processes as identical, just a different representation of the same process. (Pars 40, 42, 44)

C was describing how taking points away from one grade and adding it to the other grade was the same as the dot plot and finding the point of balance. However, it was *also* the same as having a total accumulation that was shared equally. Initially C had described the algorithm as:

> . . . it's kind of a fair, it's kind of a sharing. You know you're totaling up all that the—the total quantity and you're dividing it by the number of slots, in a sense, you would have to distribute the quantity. And so, in that sense, it's kind of a—I think of it visually in terms of leveling. (Par 20)

Here, C was describing the algorithm but stated that he was visually thinking of it as leveling. Later he described leveling-off in arithmetic terms. He said, "We're turning the process into—calculating the arithmetic mean into a series of additions and subtractions" (Par 50).

C did not feel he could articulate a full argument for connection. He could connect the algorithm to leveling-off and he could connect balancing to leveling-off but he stated his only connection was through this one way of thinking. He said:

> But, if I had to construct an argument, it would be something along the lines of, OK: division—the standard algorithm is partitioning the total into five equal parts, right? . . . That would—I guess it is more—I think leveling is more of the connecting point for me. (Pars 58, 60)

C believed that leveling-off was one way to connect the two different ways of conceiving of arithmetic mean. As I re-examined the other experts' work in the task-based interviews, I began to develop a description of how leveling-off might be a point of connection between the two conceptions of arithmetic mean.

Chapter 5

Implications

5.1 Introduction

Learning arithmetic mean has been seen as problematic in the past. An articulation of a goal understanding for students could be developed by examining experts' understanding of arithmetic mean. The purpose of this study was to explore how experts understand arithmetic mean in order to create descriptions of understanding that would be useful to researchers, curriculum developers, and teachers. The study has confirmed the importance of two previously identified conceptions: understanding the algorithm for arithmetic mean and understanding arithmetic mean as a mathematical point of balance. The study also revealed experts' lack of an articulated connection between these two understandings. In this chapter I first summarize key aspects of the experts' understandings of these two conceptions of arithmetic mean. Based on data analyzed in the previous chapter, I hypothesize an implicit connection between the experts' understanding of these two conceptions. Finally, I discuss the limitations of the present study and implications for future studies.

5.2 A Description of Understanding Arithmetic Mean

5.2.1 Algorithm

The arithmetic mean is often thought of as the numerical result of an algorithm that requires that the data values of a set are added together and then divided by the number of data points in the set. The experts of the study, however, understood the *meaning* of the operations within the algorithm as well as the nature of the result of the algorithm. The experts understood the arithmetic mean as a consequence of partitive division operating on a total accumulation of data values. The total accumulation of the data values did not change, but the value of all the data was shared equally among all data points (partitive division). The result of this division was a per-one ratio that represented what one data point would be if the total accumulation was equally shared.[8] The arithmetic mean was used as a suitable replacement for every data point in the set because the total accumulation of all the data values shared equally was represented by a per-one ratio; the arithmetic mean.

The implications of understanding arithmetic mean as an algorithm are quite varied. A question arises of how children develop understanding of the meanings of the operations involved in the algorithm. Additionally, how do children develop understanding of *any* ratio as being a per-one ratio? This development takes place only after prerequisite understandings for addition, partitive division, and relationships between numbers (ratios) have been developed. Many children being introduced to the arithmetic mean are capable of understanding total accumulation, they also may be ready to build understanding of equal sharing, and thus they appear to be ready to work with the algorithm of arithmetic mean. However, when the experts referred to the algorithm of arithmetic mean, they were referring to qualities of the result of that algorithm. Their understanding of *partitive* division allowed them to understand the nature and nuances of the result. This is not to say that arithmetic mean should not be introduced early. However, I believe it would be beneficial to develop a student's understanding of the meaning of operations in order to more deeply develop his or her understanding of

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[8] This appears to relate to what Cortina has called "normative" in recent research studies (2002)

arithmetic mean. Focusing on addition as the accumulation of values and partitive division as fair sharing of the data values would be beneficial to the development of understanding of the algorithm. The question remains as to whether understanding the operations is sufficient for further development of understanding arithmetic mean as a ratio. Once the result of the arithmetic mean algorithm is understood as a ratio, the quality of that per-one ratio as representative of the set will also need to be understood. This means that children will need experiences to help them understand per-one ratios as representative of the set (such as miles per hour as a ratio representative of speed). An additional question that remains to be examined is whether learning ratio before arithmetic mean is more useful that learning arithmetic mean and then ratio.

5.2.2 Mathematical Point of Balance

Experts understood balance as the value at which the sum of the deviations of all data points equals zero. The activity of equalizing the deviations (by summing the deviations to zero, comparing deviations over with under, or cancelling deviations) was the method for finding the point of balance. Although the experts could use what I assume was their experience to make a fairly accurate estimate, it was the mathematically precise definition of balance as equalized or cancelled deviations that they used to be sure they had obtained an exact arithmetic mean.

The experts sometimes cancelled deviations numerically after making an initial proposal for an arithmetic mean. At other times the experts used the given representations (such as a bar graph) to assist in the equalizing of the deviations; making the deviations above the proposed mean equivalent to those below. This was seen in work with the Building Height Problem and the Blood Pressure Problem. When the experts were given a data set's arithmetic mean and asked to discover other things about the data set (Class Average Problem) they also used the conception of arithmetic mean as a point of balance. In this case they used their understanding of torque (the deviations multiplied by the statistical weighting) in order to determine class size.

The implications for this conception of arithmetic mean are again related to the developmental readiness of students to conceive of balance. Mokros and Russell had as one conclusion of their study on understanding arithmetic mean as a mathematical point of balance that, "Using one poorly understood set of ideas—the physical relationship of weight and distance—may not help students understand another set of difficult ideas—the numerical relationship between the mean and the data" (Russell & Mokros, 1996, p. 361). The work done by Piaget and Inhelder as well as that done by Siegler and Chen suggests that developing an understanding of balance takes time and proceeds through several stages. The current study highlighted that one way experts understand arithmetic mean as a mathematical point of balance is in a mathematically precise way: the point at which the sum of the deviations are equal. This understanding takes time to develop separate from understanding arithmetic mean. The question remains: how do we help students develop both a physical and a mathematical understanding of balance and also relate that understanding to arithmetic mean? Does the concept of arithmetic mean as a mathematical point of balance grow out of (at least in part) understanding of the algorithm?

It was this question that was one impetus for examining connections the experts made between the two articulated understandings of arithmetic mean: the algorithm for arithmetic mean and arithmetic mean as a mathematical point of balance.

5.3 Hypothesized Connection Between Understanding the Algorithm for Arithmetic Mean and Arithmetic Mean as a Mathematical Point of Balance

The data suggest that the experts generally were unable to articulate a connection between understanding the algorithm for arithmetic mean and arithmetic mean as a point of balance. However, the data also suggest that the experts had a way of thinking about finding the mean, leveling-off, that was related to each of the two conceptions. The term “leveling-off” was not always used, but the experts gave explanations that can be described by that term.

The experts understood the algorithm for arithmetic mean as a result of partitive division. The data values were accumulated and then shared fairly with each data point. This was equated to leveling-off by suggesting that the fair sharing could be done by simply moving pieces of a bar graph (or numerical amounts) until every data point was the same value. Thus, when leveling-off was completed, the height of one bar was equated with how much value there was per-one data point.

In other tasks, the experts used leveling-off to explain how they were thinking about arithmetic mean as a mathematical point of balance. The activity of leveling-off allowed the experts to “visualize” the deviations from a proposed mean. In order to obtain balance, and thus find the arithmetic mean, the pieces over and the pieces under had to be equivalent. The experts implied that leveling-off would find a point of balance because any amount over the mean exactly matched an amount under the mean. In this sense, the leveling-off was the same as numerically cancelling deviations. When leveling-off to find a point of balance, the experts focused on the deviations from the mean and their equivalence.

Because the experts seemed to connect the notion of leveling-off with both conceptions, it was reasonable to consider the notion as a clue to one aspect of how the two conceptions may be connected for the experts. As the experts completed tasks, they moved seamlessly between the two conceptions using the conception with which it was easiest to work in the given task or representation. How did they develop this connected understanding that allowed them to work flexibly with both conceptions of arithmetic mean? The following is a hypothesis of how a connected understanding may develop in a child. It is generated from the data which suggest that leveling-off is one possible connecting strategy.

A child's earliest experiences with the concept of arithmetic mean[9] may be with two data points. In finding the arithmetic mean, the child may use a variety of ideas. They likely have experiences that include finding the midpoint between the two data points. This might be done physically, such as on a dot plot, or it may be done numerically by adding the two values and using division to share the total amount. The child may use a leveling-off strategy such as increasing one number and decreasing the other number by the same amount. Or, the child may think about the midpoint as equidistant from both values and use a primitive notion of balance. This assortment of experiences might contribute, for some learners, to a connection among notions of partitive division, leveling-off, and balance.

When the data sets become larger and more demanding, the connection among these ideas may become less obvious and therefore more difficult to make explicit. The

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[9] It is important to note that I do not that suggest the experience should be overtly named "arithmetic mean" or even "average".

data in this study suggest that the notion of leveling-off may (perhaps not explicitly) continue to serve as a connecting idea.

An early understanding of arithmetic mean is to conceive of the arithmetic mean as a fair sharing of total accumulation. This conception appears to initially involve a specific order: First find the total accumulations and then share that accumulation equally among all data points. This is an activity (often described by the operation of division) that requires a total amount. However, this can develop into understanding the activity of division without first finding the total accumulation. For example, a concrete situation such as a set of Unifix towers of different heights can be solved through leveling-off. This activity can help the child see the situation as one that calls for partitive division. Thus a connection between the strategy of leveling-off and partitive division for finding the arithmetic mean is made.

As understanding of the arithmetic mean continues to develop, the focus of the child moves from the total accumulation of the data set being equally shared to the amounts (or deviations) that data points are from an estimated mean being shared to level-off the data. It is possible to develop either the cancelling of positive and negative deviations or the equalizing of deviations from this leveling-off. Typically, a data set is given and the arithmetic mean is unknown. In order to use leveling-off, an arithmetic mean needs to be estimated. When using a proposed arithmetic mean, it is reasonable to assume the child will often have an incorrect estimated arithmetic mean. The result when the child levels-off the data is that there will be extra deviation in one direction or the other. This provides an opportunity for the child to focus on the deviations. As corrections to the estimate are made, the need for equivalent deviations becomes more apparent. Thus, leveling-off is connected to the need for equivalent deviations.

Previous research (Inhelder & Piaget,1958; Siegler, 1976) suggests that children take much time to develop a conception of balance. It is possible that the conception of physical balance as torque must be developed independently from conceptions of arithmetic mean. Once a conception of arithmetic mean as a point where the deviations are equivalent has been developed, the conception can then be connected with an equivalent conception used in physical balance. It is unclear how the mathematical and physical connection is made. However, I hypothesize that the use of representations for data sets that highlight the physical nature of balance (such as the dot plot) may encourage the connection between the physical and mathematical conceptions. Further research may fill out the proposed progression for learning of arithmetic mean and determine its efficacy.

5.4 Limitations of the Study

This study was exploratory in nature. I interviewed five people whom I identified as experts on arithmetic mean. This was based upon my perception of their abilities within their given fields (Mathematics, Statistics, Engineering). Not all fields that would rely on understanding arithmetic mean were included, thus I may have missed an understanding of arithmetic mean that is critical in one of these areas. For example, a physicist may have added another dimension to the results of the study. Additionally, because I used five participants, my results are limited. It would be beneficial for future study to examine the conceptions of a greater number of participants.

The study was limited by the effectiveness of the tasks and of the interview questions. The telephone interviews were an attempt to compensate in part for limitations of the face-to-face interviews.

5.5 Benefits of the Study

The study added further clarity to conceptions of arithmetic mean. Although both conceptions described in this study may be found in the literature, the articulations that resulted from the analyzed data clarified the nature and furthered the descriptions of these conceptions. Understanding arithmetic mean as a per-one ratio is not a typical way of thinking of arithmetic mean according to the literature, however, the experts did use this ratio understanding. The clarification of understanding arithmetic mean as a per-one ratio gives a stronger basis on which to build future research. A main benefit of the study was the hypothesized connection between the two conceptions. Previous research has indicated a difficulty in connecting the conception of the algorithm for arithmetic mean with the conception of arithmetic mean as a mathematical point of balance. The current study suggests that leveling-off is one possible connecting point. A hypothesized development of the connection was articulated in order that further research, such as teaching experiments, may be conducted to examine the efficacy of leveling-off as a way of connecting the two articulated conceptions of arithmetic mean.

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Appendix A

Researcher's Articulation of Her Own Understanding of Arithmetic Mean (April 26, 2005)

Arithmetic mean is a measure of distribution; a point of balance that recognizes individual contributions to the group yet gives the whole group a representative number. In one sense, it is a number that represents what each individual contributes if the contributions were the same (no variation), yet the accumulated amount was unchanged (leveling-off). The arithmetic mean says something about the entire distribution.

There are several ways to think about arithmetic mean. Konold and Pollatsek (2000) chose to think of mean as a "signal in a noisy process". That is, what the process might eventually settle down to. They suggested that it reduced the data due to complexity. Another way to think of arithmetic mean is as a "normalizer". As a statistical tool, it is often used to compare distributions of unequal sizes. The arithmetic mean is a ratio that is a "unit factor." What each contribution would be . . . . So that comparison is possible.

Appendix B

Interview Schedules

B.1 PHASE ONE

Interview Schedule draft (April 27, 2005)

(Participant should have read and signed consent form before beginning)

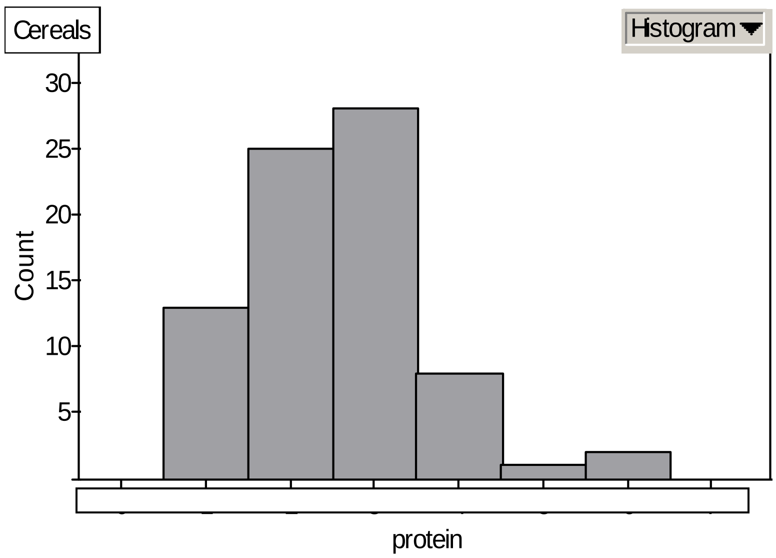
Introduction:

Good (morning/afternoon/evening). I want to thank you for your participation. First, let me give you some information about this study. The aim here is to find out more about how people understand arithmetic mean. I will be asking you some questions and also asking you to complete some tasks. You may know how to answer or do these tasks or you may not. I am not as interested in you giving me one single answer as I am in finding out about your thinking. It won't be helpful to me if you think silently and just give me an answer. It will be more helpful if you tell me what you are thinking. I will be asking for clarifications and explanations about what you have done or said. This doesn't mean that you are right or wrong. I am just trying to make sure that I understand how you are thinking about the mathematics. There are some materials that will be available for your use during the interview: you can use the calculator, the laptop computer, paper and pen. Do you have any questions before we get started?

Question 1:

Hand participant histogram on handout 1.

This data comes from the DASL website and is for free use. The histogram shown is data that is taken from the panel of 77 different breakfast cereals. The histogram shows the grams of protein in these cereals. These grams are integer amounts, but I have not shown how many grams. The count is the number of boxes of cereal that have a particular amount of grams of protein. Do you have any questions about reading this histogram? (Answer questions, if necessary)



ASK: *Can you find a point through which a vertical line would divide the graph into two pieces of equal area?* Be sure participant marks this point on the paper.

ASK: Is *there a statistical measure that this point is identified with?*

If participant answers "yes": *Which one?*

"Mean" *Would you draw a line where the median would be? Can you explain what you are doing to decide where the line belongs?*

Be sure to probe as to what characteristics of the histogram are being examined to make this decision. Probe into WHY the answer is given. Would having the number of grams revealed help you at all? How?

“Median” *Would you draw a line where the mean would be? Can you explain what you are doing to decide where the line belongs?*

Be sure to probe as to what characteristics of the histogram are being examined to make this decision. Probe into WHY the answer is given. Would having the number of grams revealed help you at all? How?

If participant answers “no”: *What would this point tell you about the data?*

***ASK**: On the computer I have a program called “Fathom”. Do you know how to get Fathom to show the mean on that graph?*

- If yes: *Please show me*
- If no: *I can show you how. Go to the graph menu and then plot value. Now under function expand “statistical” and “one-variable”. Double click on “mean”. Now click OK.*

***ASK**: Compare your answer to the one Fathom generated. Does Fathom’s determination of the location of the mean make sense to you? How so? Or Why not?*

Possible answers and follow ups:

- Skewness: *What about skewness makes the answers different? What if the data were skewed in the other direction? What if the data were not skewed?*
- Found average of the range: *Can you show me what you mean using some numbers? Why would this be different from Fathom’s answer?*

- Did not take each height into account: *What would you have done differently to take the heights into account? What did those heights represent?*

ASK: *Moving as few points as possible, how could you alter the data set to make the median to be to right of the mean? Why did you choose those points?*

Question 2:

Give participant a blank sheet of paper and pen.

ASK: *If there were no such thing as arithmetic mean, would you invent it? Why?* After participant has finished, probe for factors they find valuable and why.

ASK*:* *In (your area of expertise), is there a particular aspect of arithmetic mean that your students find difficult? Why do you think that is?* Probe carefully as to how they view the problem situation, what they might see as a remedy and why.

ASK: *Without using the algorithm, how would you define arithmetic mean? Why did you include _____ in your definition?*

Question 3:

Handout with following situation:

In a certain class there are more than 20 and fewer than 40 students. On a recent test the average passing mark was 75. The average failing mark was 48 and the class average was 66. The teacher then raised every grade 5 points. As a result the average passing mark became 77 1/2 and the average failing mark became 45. If 65 is the established minimum for passing, how many students had their grades changed from failing to passing?

ASK: *Please read this situation. Do you have any questions about the situation? Would you please explain what you are doing as you solve this problem.*

As participant answers – probe as to why he/she is doing certain steps, what makes that important. At the end, probe to see if he/she has another way or ideas of how to solve that are more "proficient".

Question 4:

In the most recent issue of Mathematics Teacher, I read about an interesting mathematical relationship (Brown and Rizzardi, 2005). I'd like you to look at problems similar to the ones discussed and solve the problems as the authors suggest for the students to do.

Give Handout with the following Problems:

1. Two people drive from their hometown to a city that is 150 miles away. The first person is cautious and drives 50 mph. The second driver is a bit more reckless and drives 70 mph. Find the average speed.
2. My sister and I drove across the country and decided we would share the driving time equally. Every four hours we would stop to buy gas and coffee and trade driving. My sister started and was very excited, she drove 70 mph for the first

four hours. When it was my turn we were headed up a lot of hills and I drove 50 mph for my four hours. What was the average speed for the eight hours?

3. Pat was training for a long run and decided to incorporate a particular training segment into his workout. He would run up a 1/2 mile long hill and then run down the hill for that segment. Running uphill, he ran at a pace of 10 minutes per mile. Running downhill, he ran at a pace of 8 minutes per mile. What was the average speed of his running during that segment of his workout?
4. The math education group decided to raise some money for charity by holding an "endurance" run. Two people would run for the first four hours, another two people for the next four hours and so on. During the first four hours, the two runners decided to trade off running every hour. Runner 1 ran one hour at the pace of 10 minutes per mile. Runner 2 then took over and ran at a pace of 8 minutes per mile for the next hour. What was the runners' average speed at the end of the first two hours?

ASK: *(As participants answer) Why did you choose to solve it that way? What cues were you using to help you?*

There is a possibility that harmonic mean will be used – if so, the next question can be skipped and the probing used later can be conducted here.

In the article, the authors wrote that you could use harmonic mean to find some of these averages. Are you familiar with the harmonic mean?

If NOT: Write equation for harmonic mean on paper for participant to read and analyze.

If they do know, continue.

Here is the chart the authors developed to show when you could use harmonic mean and when you should use arithmetic mean.

Give Handout:

	Equal Distance	Equal Time
Distance – Oriented: <u>Total Distance</u> Total Time	Harmonic Mean	Arithmetic Mean
Time-Oriented: <u>Total Time</u> Total Distance	Arithmetic Mean	Harmonic Mean

ASK*: Do you understand this table? Can you relate each of the four problems given to one section of the table? How?* Once you are sure they have understanding of the situation, continue.

ASK: *The authors write that this relationship is true, but they do not explain WHY it is true. Could you explain why this particular relationship holds without using algebra?* PROBE

Thank you for your help in my dissertation study. If you have any questions, you can best contact me by e-mail at the address given on the consent form. Also, I may be asking you to respond to a written description of your understanding in the future.

B.2 PHASE TWO

Interview Schedule – changed July 19, 2005

(Participant should have read and signed consent form before beginning)

Introduction:

Good (morning/afternoon/evening). I want to thank you for your participation. First, let me give you some information about this study. The aim is to find out more about how people understand arithmetic mean. I will be asking you some questions and also asking you to complete some tasks. You may know how to answer or do these tasks or you may not. I am not as interested in you giving me one single answer as I am in finding out about your thinking. It won't be helpful to me if you think silently and just give me an answer. It will be more helpful if you tell me what you are thinking. I will be asking for clarifications and explanations about what you are thinking while you are working. This doesn't mean that you are right or wrong. I am just trying to make sure that I understand how you are thinking about the mathematics. There are some materials that will be available for your use during the interview: you can use the calculator, the laptop computer, paper and pen. Do you have any questions before we get started?

Question 1:

Hand participant histogram on handout 1.

This data comes from the DASL website and is for free use. The histogram shown is data that is taken from the panel of 77 different breakfast cereals. The histogram shows the grams of protein in these cereals. These grams are ***integer*** *amounts, but I have not shown how many grams. The count is the number of boxes of cereal that have a* *particular amount of grams of protein. Do you have any questions about reading this histogram?* (Answer questions, if necessary)

SAY: *I have placed a vertical line on the graph that divides the graph into two pieces of equal area. Do you have any questions as to how I placed the line or do you disagree with the placement for any reason?* (Respond to any questions if necessary – the goal of this is to be sure that the participant is reading the histogram correctly.)

ASK: Is *there a statistical measure that this point is identified with?*

If participant answers "yes": *Which one?*

"Mean" *Would you draw a line where the median would be? Can you explain what you are doing to decide where the line belongs?*

Probe into WHY the answer is given and how it was arrived at. How is the given information being used to determine mean/median? Would having the number of grams revealed help you at all? How?

"Median" *Would you draw a line where the mean would be? Can you explain what you are doing to decide where the line belongs?*

Probe into WHY the answer is given and how it was arrived at. How is the given information being used to determine mean/median? Would having the number of grams revealed help you at all? How?

If participant answers "no": *What would this point tell you about the data? Could you locate the mean with the given information? How?*

Question 2:

Give participant a blank sheet of paper and pen.

ASK: *If there were no such thing as arithmetic mean, would you invent it?* [Ask this first and pause for answer] *Why?* After participant has finished, probe for factors they find valuable and why.

ASK*:* *In (your area of expertise), is there a particular aspect of arithmetic mean that your students find difficult? Why do you think that is?* Probe carefully as to how they view the problem situation, what they might see as a remedy and why.

ASK: *Without using the algorithm, how would you define arithmetic mean? Why did you include _____ in your definition? Or Why is ____ important for your definition?*

Question 3:

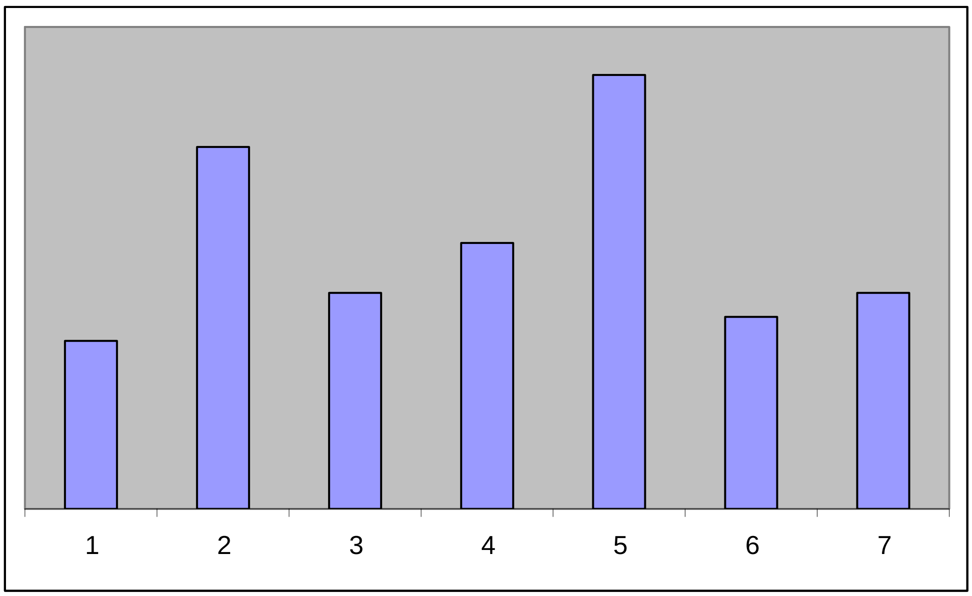
Handout with following question: A class of students took a test. The class average on the test was 68. The average grade of the students who passed was 80 and the average grade of the students who failed was 64. What percentage of the class passed?

ASK: *Please read this situation. Do you have any questions about the situation? Without using algebra, would you solve this problem and explain to me how you are solving it?*

As participant answers – probe as to why he/she is doing certain steps, what makes that important. At the end, probe to see if he/she has other ways or ideas of how to solve that are more "proficient".

Question 4:

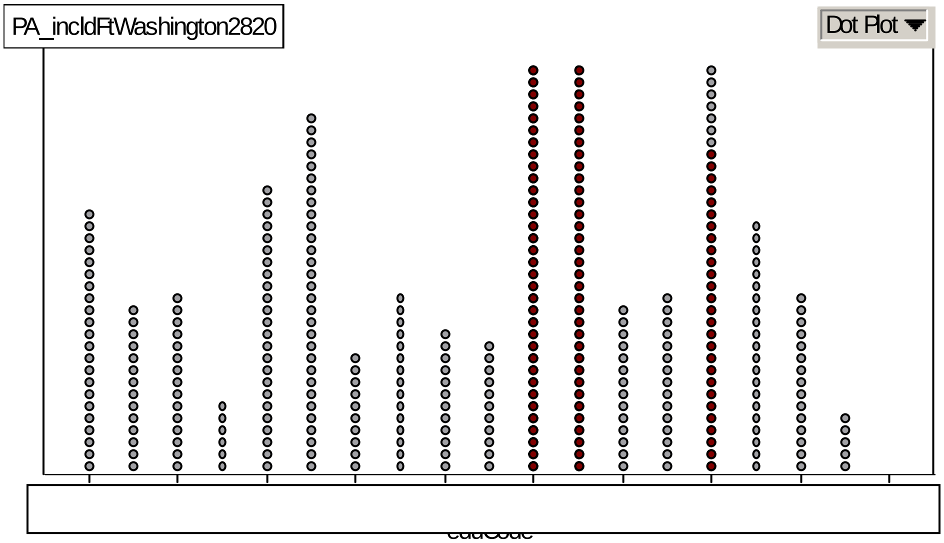
Handout: Suppose the following picture is depicting building heights. What would the average height of the six buildings be? PROBE: *How are you deciding on that height? Are you sure that it is accurate, why?*



PROBE: How does this relate to the definition of mean you gave me earlier? In the way you did this, where are you using _____ that you used in earlier work with am?

Question 5:

Give participant the handout of distribution above a horizontal line (dot plot).



ASK: *This is a distribution of a large amount of data. The mean of the data is shown. This was created in Fathom, so the red dots mean that there is more than one data point being represented by the dot. If I added a data point here (mark 2 units to the right), how would that affect the mean?*

Probe: *How do you know it will move it in that direction? Do you know by how much? Why or Why not?*

ASK: *Now suppose of instead of the data point being added here (2 units to the right), suppose it was added here (mark 4 units to the right). Now, how would that affect the mean?*

Probe: *How do you know it will be impacted in that direction? Do you know by how much it will change? Do you know by how much in comparison to if it were this other point (2 units to the right)? How do you know that?*

ASK*: In the above problem you used ______ to find the mean, but here you used _______ to change the mean. According to you, these both represent the mean. How then, are they related?*

Thank you for your help in my dissertation study. If you have any questions, you can best contact me by e-mail at the address given on the consent form. Also, I may be asking you to respond to a written description of your understanding in the future.

IF there is time I could use the optional question on Teacher Grades:

Give participant the copy of the excel file as shown.

HW-9	HW-5	Quiz -16	HW-5	Test - 42	HW-8	Quiz -20	Final - 63	Total - 168	GRADE
8	5	15	4	40	8	18	58	156	92.86
7	5	14	5	38	7	16	55	147	87.50
9	5	13	5	41	8	18	50	149	88.69
5	4	11	3	33	8	16	62	142	84.52

HW	HW	Quiz	HW	Test	HW	Quiz	Final	GRADE
88.89	100	93.75	80	95.24	100	90	92.06	92.49
77.78	100	87.50	100	90.48	87.5	80	87.30	88.82
100.00	100	81.25	100	97.62	100	90	79.37	93.53
55.56	80	68.75	60	78.57	100	80	98.41	77.66

ASK: *The top file shows the grade by points earned out of points possible and the final grade was figured from total points earned out of total points possible. In the lower file, each grade the students earned was converted to a percentage and the final grade was figured by finding the average percent grade for the eight graded assignments. Why do Student 3 and 4 have such different grades?*

PROBE: *How do you know this is the cause of the difference? What are you using to help you make this determination?*

Interview Schedule – changed June 11, 2006

Participant should have read and signed consent form before beginning)

Introduction:

Good (morning/afternoon/evening). I want to thank you for your participation. First, let me give you some information about this study. The aim is to find out more about how people understand arithmetic mean. I will be asking you some questions and also asking you to complete some tasks. You may know how to answer or do these tasks or you may not. I am not as interested in you giving me one single answer as I am in finding out about your thinking. It won't be helpful to me if you think silently and just give me an answer. It will be more helpful if you tell me what you are thinking. I will be asking for clarifications and explanations about what you are thinking while you are working. This doesn't mean that you are right or wrong. I am just trying to make sure that I understand how you are thinking about the mathematics. There are some materials that will be available for your use during the interview: you can use the calculator, the laptop computer, paper and pen. Do you have any questions before we get started?

Question 1:

Give handout 1 (shown below) to participant.

Assignment 1	90
Assignment 2	73
Assignment 3	80
Assignment 4	85
Assignment 5	97

SAY: *This sheet shows five grades of a student. How would you find the average grade?*

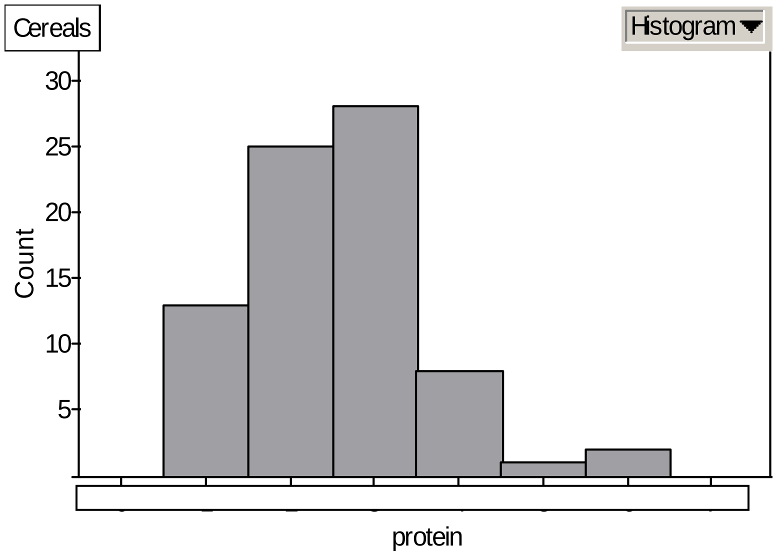
Probes: *If they start with the algorithm, ask them not to do it right away. Instead ask them if there is another way to find it. Why are you doing . . . ? How does that help you find the average? What does this number mean that you got? (Ask about each step and why it was appropriate for finding the average based on their description of what average means).*

Question 2:

Hand participant histogram on handout 2.

This data comes from the DASL website and is for free use. The histogram shown is data that is taken from the panel of 77 different breakfast cereals. The histogram shows the grams of protein in these cereals. These grams are ***integer*** *amounts that are uniformly spaced, but I have not shown how many grams. The count is the number of boxes of cereal that have a particular amount of grams of protein. Do you have any questions about reading this histogram?* (Answer questions, if necessary)

Have them interpret this graph first.



SAY: *Would you draw a vertical line where the mean would be? Please explain what you are doing to decide where the line belongs.*

Probes: *What in the diagram did you use to help you decide on this placement? What were you looking at? How are you thinking about this?*

Question 3:

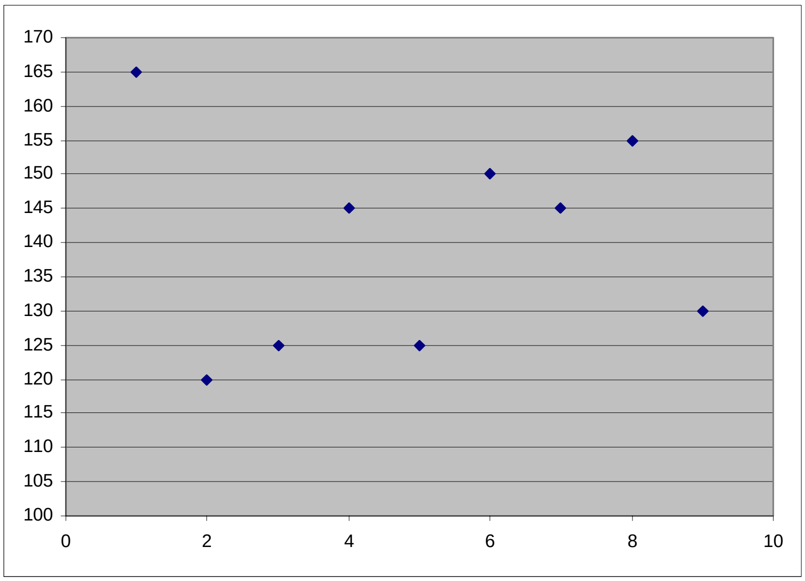
Give participant chart on handout 3 with blood pressure numbers.

SAY: *I have plotted nine different blood pressures (the top number) of a patient. What do you think the average blood pressure is for this patient? What are you thinking/doing to figure out what that average is?*

Probes: If participant is using algorithm, ask if there is anything about the chart that might help them. What is it? How is it helpful?

Cover all but two dots

ASK: How would you find the average of just these two? Why are you doing this? If participant equalizes deviations – reveal one more dot and ask how they would find the average of the three? How is what you did different? The same?



Question 4:

Handout with following question: A class of students took a test. The class average on the test was 68. The average grade of the students who passed was 80 and the average grade of the students who failed was 64. What percentage of the class passed?

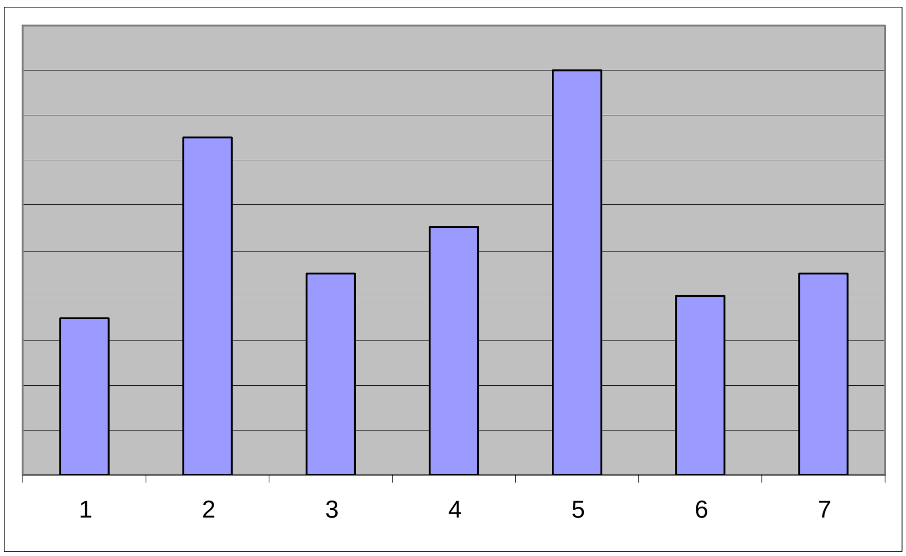
ASK: *Please read this situation. Do you have any questions about the situation? Without using algebra, would you solve this problem and explain to me how you are solving it?*

As participant answers – probe as to why he/she is doing certain steps, what makes that important. At the end, probe to see if he/she has other ways or ideas of how to solve that are more "proficient".

Question 5:

Handout: Suppose the following picture is depicting building heights. What would the average height of the seven buildings be?

PROBE: *How are you deciding on that height? Are you sure that it is accurate, why?*



PROBE: How does this relate to the definition of mean you gave me earlier? In the way you did this, where are you using ____ that you used in earlier work with am?

Question 6:

ASK: *When I looked up the definition of arithmetic mean in a mathematics dictionary, the definition was the algorithm: add up all the values and divide by how many data points you have. In this interview, you have been using the idea that arithmetic mean is a mathematical point of balance. How do you determine the arithmetic mean is a* *mathematical point of balance from this standard algorithmic definition? Why should we expect this to be a mathematical way to represent this definition?*

PROBE: (Be sure to keep aware of whether there is a quick answer or something is being made up). How would you explain to a 6th grader how the algorithm is related to this idea of balance?

Thank you for your help in my dissertation study. If you have any questions, you can best contact me by e-mail at the address given on the consent form. Also, I may be asking you to respond to a written description of your understanding in the future.

B.3 PHASE THREE

Telephone Interview Schedule – December 14, 2006

Thank you for agreeing to talk with me one more time. ***May I tape our conversation?***

About a year and a half ago I interviewed you and asked you to complete some tasks that were related to finding the arithmetic mean. I've been working on analyzing what I can learn from the data I collected. As I've worked I have recognized that there are two different ways that you worked with the arithmetic mean.

One way was to use the algorithm for arithmetic mean – or at least the operations within the algorithm. That is you did things related to adding up all of the values of the data points and dividing this total number by the number of data points you had.

The second way you worked with the arithmetic mean was as a mathematical point of balance. That is, you found a place in the data that was as if you were balancing out all of the values of the data.

Does that sound like a fair description of what you remember?

I would like to ask you a few questions about these two approaches.

[If needed when gathering answers: Could you explain this without using algebra or variables as replacements when you are giving your explanation?]

Could you explain how you understand that the arithmetic mean defined by the standard algorithm and the mean as the mathematical point of balance are the same thing?

Rephrases:

If I'm looking at the standard algorithm, what about it suggests that the quantity that I am finding will be the point of balance?

Could you explain how you understand arithmetic mean defined by the standard algorithm and the mean as the point at which the deviations are total to zero are the same thing?

***How could you explain to a 6th grader how the algorithm is related to this idea of balance*?**

We seem to start learning about arithmetic mean as a result of the algorithm. How do you think this develops into an understanding of arithmetic mean as a mathematical point of balance?

KEY PROBE: GET A HANDLE ON WHETHER THE SUBJECT HAS CLEAR ANSWERS TO THESE QUESTIONS ALREADY OR IS TRYING TO COME UP WITH ANSWERS BECAUSE OF YOUR QUESTIONS. IF THEY ARE COMING UP WITH ANSWERS, MAKE SURE THAT YOU HAVE GOTTEN EVERYTHING THEY TYPICALLY THINK ABOUT BEFORE LETTING THEM CONTINUE!

Again, I thank you for allowing me to do this follow-up question. If you have any questions or concerns, you can always contact me. My information is on the Informed Consent form I gave you and/or I can email the information to you again.

Thanks.

Appendix C

Interview Transcripts

C.1 Task-Based Interviews

C.1.1 Partial Transcript from Task-Based Interview with C

41. I: OK. Now, you said the median would be here, whatever this value is. Um, where would the mean be for this information? The arithmetic mean?

42. P: [pause] Well, it's going to be to the left of that median, I believe.

43. I: Which median? I'm not -, in a sense you have two medians -

44. P: Yes, um it's going to be to the left of this [points to median that is the midpoint of the bar, not vertical area divider].

45. I: OK and why is that?

46. P: Well, I'm thinking in terms [places pen vertically on graph on the third bar from the left] of a balance, um. It's an interesting connection here, I think. I'll talk through it, I haven't thought my way through it, so this might be a segment of the interview you ignore, or it might be the most interesting, who knows?

47. I: [laughs]

48. P: OK, so if I start thinking about where do I need to balance this, where would this balance, in other words? Where could I put a fulcrum that would level these out? Um, assuming that this scale, using kind of standard histogram practices are, is uniform.

49. I: It is. You can make that assumption.

50. P: OK. [long pause] then, I'm kind of leaning towards, I'm not sure of why this is, but I'm kind of leaning towards [draws an arrow pointing to the vertical line originally drawn to show the division of equal area], I don't know why this is – it's not obvious to me. But, I'm kind of leaning towards that being, well not that [now points to line drawn slightly in to the left of the third bar], but this wherever I would divide it at 1/56th of the way across there. The actual mean.

51. I: Can you explain to me why that, why you're leaning that way?

52. P: Um, Well that's why I was hesitating before. I wasn't sure that I could, could explain it. Which makes me think that I need to back up, and just kind of ignore this [points to line drawn in the bar] for a second and think about where the mean is.

53. I: OK

54. P: OK, so I've got thirteen times whatever this value is [using pen to point to midpoint of first bar on the left]. We'll call it -. We could call it one, we

$$\begin{array}{l} 13\cdot x \\ 25\,(x+1) \\ 28\,(x+2) \\ 8\,(x+3) \\ 1\,(x+4) \\ \underline{2\,(x+5)} \\ \quad 77 \end{array}$$

Figure 2

could call it x, whatever, since these are going to be uniform, but let's just call it x. And we've got 25 times, ah x + 1, or I'm going to let that one represent —And we've got 28 times x + 2, and what did I call this, seven or eight [referring to the fourth bar from left]? I called it eight eventually. 8 times x + 3 [continues writing -.] so I could add all those up and divide by whatever the sum of these numbers are, lets see, what —it's 77. [Figure 2 – what is written as he is speaking]

Nothing like a little unusual way to write that. So, [26 second pause] let's do it this way, because I think this will help make this connection, if there is one. Instead of saying this is x [points to midpoint of first bar on the left], let's say, um, let's do it from this point right here [points to vertical line drawn to divide into two equal areas], where these two [points between midpoints of second bar from left and third bar from left] it would be the midpoint of these two integer values. So we have a distance here [draws horizontal line lightly from midpoint of first bar on the left to the vertical line dividing the areas] of one and a half, a distance here of half [points to midpoint of second bar from left]. We'll call those negative [draws parenthesis around values and adds negative sign – shown in figure below]. And then we have 28 times a half, 8 times -one and a half and 2 times two and a half. And that sum [draws line under this list shown, figure 3] should be zero.

$$\begin{array}{l} 13\,(-1.5) \\ 25\,(-0.5) \\ 28\,(0.5) \\ 8\,(1.5) \\ \underline{2\,(2.5)} \end{array}$$

Figure 3

55. I: And why is that?

56. P: Well it’s the —if this is the mean [points to vertical line that divides the two areas], the sum will be zero. Because it’s the deviations.

57. I: OK, so this one; 13 times negative 1 and a half, you’re saying that’s the deviation from here to this point that you’re saying is the proposed mean [uses fingers to span the gap between the midpoint of first bar to the line that divided the area vertically, now the proposed mean]. Then you have 25 is negative a half, 28 is positive a half, 8 -, is there a reason you skipped that one [points to the fifth bar from the left]?

58. P: Oh, Careless [squeezes in missing information, figure 4].

$13(-1.5)$

$25(-0.5)$

$28(0.5)$

$8(1.5)$

$1(2.5)$

$\underline{2(3.5)}$

Figure 4

59. I: OK, all right.

60. P: Is that right, one and half, two and a half, three and a half.

61. I: Feel free to use the calculator if you want to, or not.

62. P: Well, I think I’m going to do it kind of by this cancelling out which is related to what I was doing before or it feels like the same thing.

63. I: OK

64. P: So we have the net here [crosses out 25(-0.5) and 28(0.5) and writes 3(0.5) to the left side] is three and the difference between what I was doing before and what I’m doing now is the weights of these [points to the histogram bars] that are further away from this proposed mean [points to line of proposed mean] are greater than when I was doing the equal areas, the weight of anything on the left side of that [circles in air above left side of histogram] will be equal to the weight of anything on the right side of that [circles in air above right side of histogram].

65. I: OK

66. P: In other words, I could have cancelled one of these blocks [waves pen above second bar from the left], one of these units, serials out with this one right

here [points to bar that is one unit high, the fifth from the left] and that's the same thing when you're dealing with median. But when you're dealing with mean it's not.

67. I: Why is that? Why is it different?

68. P: Well, I'd love to give an eloquent, mathematical statement here.

69. I: Just give the statement you're thinking of.

70. P: Well, the distance you can think of it in terms of the median, which side of the median you're on is all that matters when you're determining where the median is. Which is kind of circular, but I have a way to deal with the circularity here [points to the histogram].

71. I: OK, well, then I'll let you go back to that if you'd like.

72. P: But, when you're dealing with the mean, you are -, the distance matters, in addition to which side of the mean you're on.

73. I: OK

74. P: So, in other words, when I was doing the median [hides all of the numbers that are in parenthesis on the right side so that only the count shows] I was like, OK these are eleven, I've got thirteen here, cancel out -. But now I've got to account for these, the greater distance. So, I've got eight here [crosses out 8(1.5) and 13 (-1.5) and writes 5(-1.5) to the left], that leaves me with a net of five, negative. And then, let's see, I'm just going to add these [has pen over 1(2.5) and 2(3.5)] up to see. We'll let's see, these [crosses out 3(.5) and changes 5(-1.5) to 4(-1.5)], knock this down to a four, um, and then seven, nine and a half [pen is over 1(2.5) and 2(3.5)]. Four of those would be nine [points to 4(-1.5). Hmm, interesting.

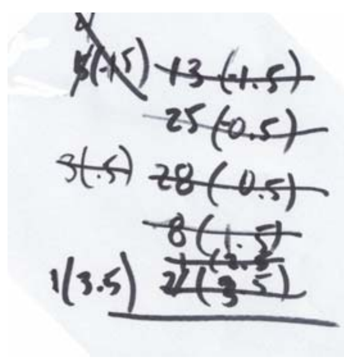


Figure 5

75. I: Four would be nine?

76. P: Four —no six, duh. Good, I'm glad it wasn't nine.

77. I: Why's that?

78. P: Cause, it was going to negate what I was saying about the differences between them -[laughs]

79. I: OK [laughing]

80. P: So this is six. That leaves me with one positive 3.5, left right? That [crosses out 1(2.5), crosses out 2(3.5) and writes 1(3.5) to the left] and one of those. OK. So, we have, this was our proposed mean [points to line on histogram]. So the sum of the deviations from that point is 3.5. If we divide that by 77 and it add it to that, that's where the mean is.

81. I: And why do I have to divide it by 77?

82. P: Um, [long pause as he holds head in his hands, then leans back] this is the way I used to average grades in my head and—select a value that I thought was close enough and then do the sum the deviations. And that remainder, you have to divide by the total number because that has to be shared among all the values. Which then can come —I mean, I don't really have the connection, I have, I need to think a little bit more. I can say that it needs to be shared but -I haven't thought through exactly why these are not shared [waves pen over calculations just done with deviations]. I mean these are basically each of these deviations is attached to a value, so there is no sharing involved, it is total value. But the sum of all of them is [pause]. Let me think a second. I'm trying to think of representing the mean another way to make this easier to explain.

83. I: OK

84. P: So, if I was thinking of leveling here [lays pen horizontally across the histogram]. Well, they're drawn the wrong way for leveling but, forget that representation, so -that's a good way to think of it. I'm basically, I'm basically leveling. So like, let's put my protein values over here [draws a vertical line and labels it protein, figure 6]. So I'm basically making, what do I have to do to each value [runs pen along deviation calculations] to get it to this value [points to proposed mean on the histogram]. This is my proposed mean [from the vertical line labeled protein he draws a dotted horizontal line across]. So, I've got some values that don't reach it, I've got others that go above it [draws a bar that is below the dotted horizontal line and a bar that

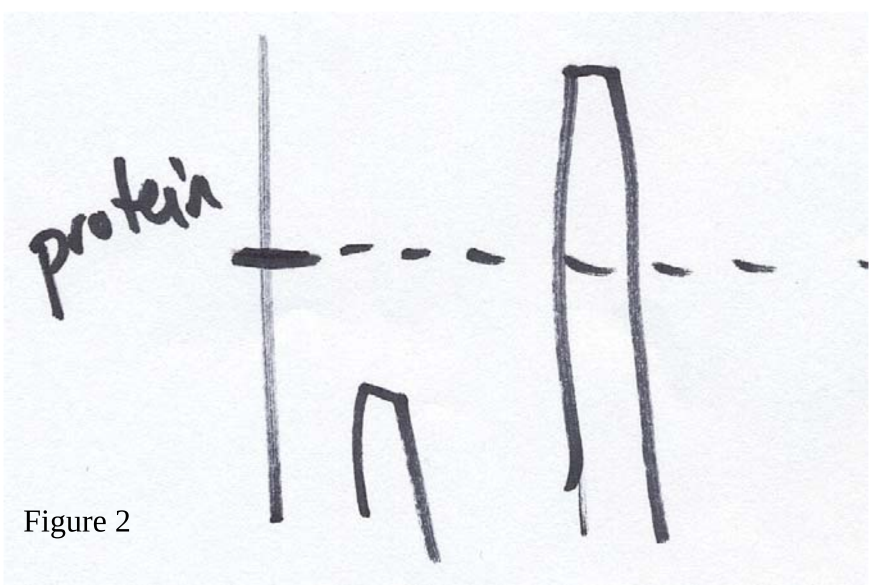


Figure 2

goes above the horizontal dotted line – see figure 7], so this process [runs pen down the deviation calculations] is basically a leveling process, relative to some arbitrarily chosen value [points to proposed mean on the histogram]. So, I'm short, I'm doing it in short hand a little bit because I'm dealing with weights rather than dealing with each one individually and cancelling it. So, this is basically a leveling process for some proposed mean but when I'm left, I've got one, I've got [gets a new sheet of paper and redraws the above figure with three bars that come up to the dotted line and one bar that is over the dotted line, figure 7] —after I've kind of done this of cancelling out process, then basically I've gotten all my values level to this, except I've got one value now that is 3.5 above [labels section of bar above the dotted line as 3.5]. So, I've got to share this [points to bar above the dotted line and then takes hand and stretches along dotted line] among all 77 values to get the actual mean. 'Cause this [points to dotted line] is not the mean.

85. I: All right, so took a proposed mean and when you share that, where would your new mean be then?

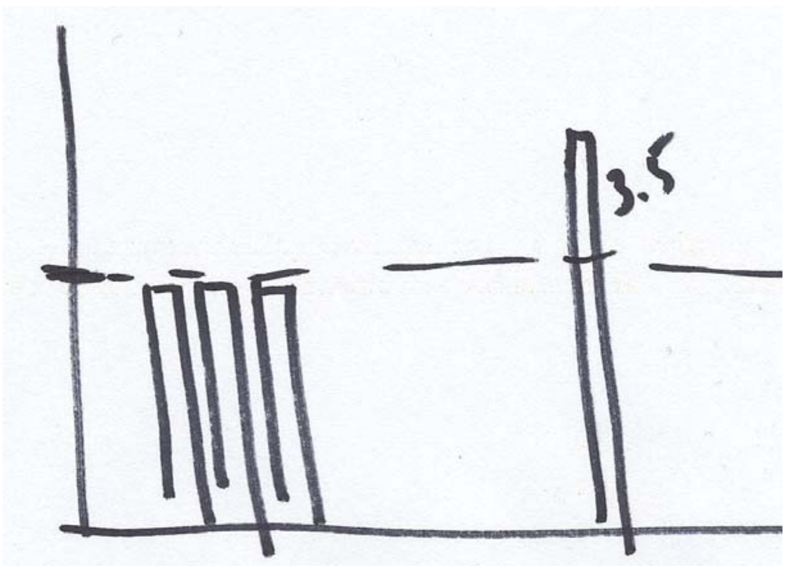


Figure 3

86. P: Well, slightly above this [pointing to dotted line].

87. I: OK

88. P: 3.5 -77th's above that.

89. I: OK – good answer. Uh, where would that be approximately—[points to histogram]?

90. P: So, it's going to be in here [pointing inside the third bar just slightly from the left], 3.5—77ths of a unit, of this unit [places pen from midpoint of bar two to midpoint of bar three].

⋮

108.P: [chuckles – pause] Yeah, I would invent it—because we call it "P___'s mean" [laughs]. No, I would invent it because it gives you an indication of a theoretical fair sharing, or equal sharing of a group of items.

109.I: Why would that be useful?

110.P: Being interested in fairness [laughs]—I mean this is the first thing that comes to mind, it's kind of amusing, but in a sense its, well if everything, if everybody got the same amount, this is what we should be shooting for. So it's kind of a measure of "all right", now you have a measure to understand how far you are from that ideal equal sharing.

111.I: Hmm [pause] How would—how would that "ideal equal sharing" be different say than, um, using the median?

112.P: [pause] Well, [pause] with the mean, you are, as I was saying a minute a go, you know, you're giving weight to—let's say this is your mean [draws a vertical line on the page]. You are giving more weight to your values depending on where they are [draws a dotted line to the left of the mean and points to the far left area], how far they are away from this mean [points to the vertical line labeled mean, figure 8].

Figure 4

113.I: OK

114.P: So, there's kind of a component of mean where it's already giving you that measure of kind of how good you are doing, so to speak -, because it is [draws an arrow up to the middle of the dotted line] dependent on this distance. The thing I don't like about that explanation is it feels sort of circular in the sense that, ah, I'm starting with the mean.

115.I: Instead of starting with -?

116.P: Just your data points.

117.I: Just data points.

118.P: But if you want to talk about the mean as the point defined such that, we can define it as that “weighted-ness”, in terms of this weight, or the distance relative to that point [uses pen to point to dotted line drawn]. You could define it as the point where the sum of all those distances is equal to zero. Which is kind of akin to the leveling process.

119.I: OK. You were a teacher for awhile.

120.P: Mm-hmm

121.I: Was there an aspect of arithmetic mean that you found your students struggled with? That you found they didn’t understand as well?

122.P: [pause] Well, there were a lot of assumptions—a lot of times where I was assuming particular understandings of mean that they didn’t necessarily have. Mean, to them, was the formula; add all the values and divide by the total number of values. That was the mean and that’s all it was. And as far as changing the way they thought of the mean or whether you were thinking of it as a balance point or point of equal, theoretical equal sharing; those different kinds of representations or conceptions of the mean didn’t, they were non-existent. Because if you went to a weighted mean, where you had multiple values, multiple data points that were the same value, the idea of multiplying those and treating those values as weights was foreign to them. I also had experience with a grade book program trying to teach teachers; help them understand the weighting process that you could use in the grade book program. They were tied to thinking the sum of your weights had to be 100 percent. And they didn’t understand the relative importance that was being placed on things. If you wanted this to count twice as much as this, then you just had to use a 2 to 1 ratio, it could be 5 to 10, it could be 2 to 1, I mean 10 to 5 or 2 to 1, whatever; as long as it reduced to that. But what changed is those relative weightings to other values. So if you had this is 2 to 1 and you add another one and that one and that one were equal weighting, because you really want that. So that process was a bit, that was difficult for teachers to get a hold of.

123.I: Why do you think that is?

124.P: I never quite really put it together at that time, so I need a little think time. I never really understood; if I felt like I understood how, why that was difficult for them, I could explain it more effectively, to them. But I was never able to explain it more effectively. I mean, these are non-mathematical folks. They understood the idea of percentages, so they understood the weighting relative to 100 percent. So maybe, the issue is not having a conception of what the whole is. Like on this problem [points to bar graph made for the first problem] where you're dealing with 77 different entries. You don't have this quick and easy sense of Ok, how many, if I want this to count 25 percent. How many of those, what's the weighting need to be relative to these other guys; the values that I have. And plus your changing two things, I guess. When you're changing the weighting, not only are you changing the multiplier, you're changing what you're dividing by. It's co-variational thing that's going on there. If you have your quiz grades and you have your homework grades and you have your test grades [writes these words down the side of a paper], if you say, Oh OK, I want this to count twice [writes a 2 next to "test"] and this to count once [writes a 1 next to "hw"] and this to count once [writes a 1 next to "quiz"]. Then you say, "well, maybe I want this to count three times" [crosses out 2 next to test and writes a 3]; all you, the thing that's real easy is to say this is counted three times as much as the homework, but in terms of the proportion of your percent of your overall grade, we've gone from a half to 3/5ths [writes ½ →3/5]. So, that kind of dual change – they never really got this conception of how it was relative to whole overall average

125.I: Without using the algorithm, how would you define the arithmetic mean?

126.P: I can think of defining it multiple ways. Do you want me to decide which one is?

127.I: Tell me all of them if you want.

128.P: Well, I think of already said a couple of them, but you could define it as the point at which the sum of the deviations was zero [points to vertical line drawn and labeled "mean" and then points to the left to the dotted line].

129.I: OK

130.P: Meaning the deviations of the data points from that value. You could define it as the point of theoretical equal sharing, in terms of the leveling [points to drawing above of bar graph where leveling was described]. It's essentially the, I think, I want to call those that it's essentially the same thing; just one's a more technical definition, you know how do you get it to. One's a little more informative about how you might go about finding it than the other.

131.I: OK. Which one's more informative?

132.P: Well

133.I: About how you go about finding it?

134.P: I guess in a numerical sense, the idea of—or maybe mathematical formality, maybe it's not any easier. The idea of looking for where the sum of the deviations from that point is going to be zero—that's not going to be any easier, necessarily. Because, you're going to be thinking about moving this around [lays pen vertically over line labeled "mean" and moves pen slightly to left and right], you know, to figure that out. And as soon as you move it, it changes all of the deviations where the leveling, if you think of, if you plot all the values like this [waves pen above the bars drawn on the graph above] and you just think of chopping off units here [points to bar that is labeled 3.5 higher] and adding them to those that are below that [points to bars that are uniform and at proposed mean]. A third grader could do that, maybe even a first grader. [laughs] You know you could define it as the balance point. Which again is related to this sum of the deviations being zero [points to the line labeled "mean" and the dotted line to the left].

135.I: How is that related?

136.P: Well, I think I did it.

137.I: Pull out this sheet [gives P handout: one with the histogram]?

138.P: Yeah. Where did I do -? [looks at both sheets] Didn't I draw something else?

139.I: I think you have everything you've drawn.

140.P: Maybe I was thinking of this picture [vertical line labeled "mean" with dotted line to the left].

141.I: OK

142.P: Your balance, just from a physics stand point. Your balance is going to be where your torque is zero. So you've got some values pushing this way [uses arms held in the air to describe – pushes down with right hand] and you've got other values pushing this way [pushes down with left hand with hands facing each other] and it's where those are going eq. . .cancel each other out. So it's essentially this process. This process of cancelling out [runs pen over computation that was done with histogram as shown below, figure 9] that torque depends on difference from the mean, distance from the fulcrum and the amount there. [Interviewer sneezes for third time] Stop it.

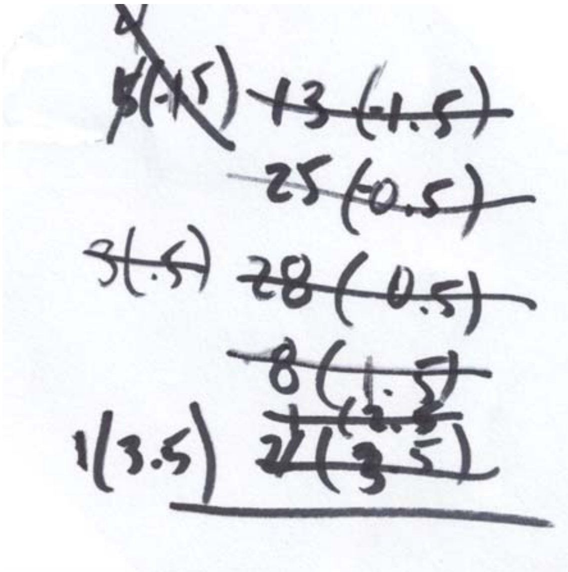


Figure 5

143.I: Excuse me. Sorry about that

144.P: So, that would be another way to think about what I was doing here. Because I'm cancelling out these weights [again waving pen over the computation shown above].

C.1.2 Partial Transcript from Task Based Interview with J

9. I: OK, let's do this. [places histogram of cereal in front of P] I got this from the DASL website, if you've ever heard of the DASL website— [Figure 1]

10. P: No

11. I: It's a website that has a number of different data collections on it. And, it's for free use, which is really nice and this is created with, um, Key Curriculum Press has a program out called "Fathom" and this was a graph that was created with it. This is a histogram and it's a histogram of the number of grams of protein in 77 different cereal boxes.

12. P: OK

13. I: OK, so on the [points to vertical axis] vertical we have the number of boxes and across the horizontal we have the number of protein grams and I've hidden how

many in each one, but there is an integer value for each of these [points to the bars of the histogram].

14. P: OK

15. I: So there are no non-integer values, it's uh, oh its 5 grams or 8 grams. That's how we're working with this. What I've also done is I've placed a dotted vertical line [points to red dashed line on histogram –figure 1] in a place where it approximately splits the area of these histograms in half.

16. P: OK

17. I: My first question to you is, is there a statistical measure that you could relate to that vertical line, that dotted line?

18. P: It would be the median.

19. I: OK. And why do you say that that particular point is the median?

20. P: Precisely because it splits the sample into two equal parts.

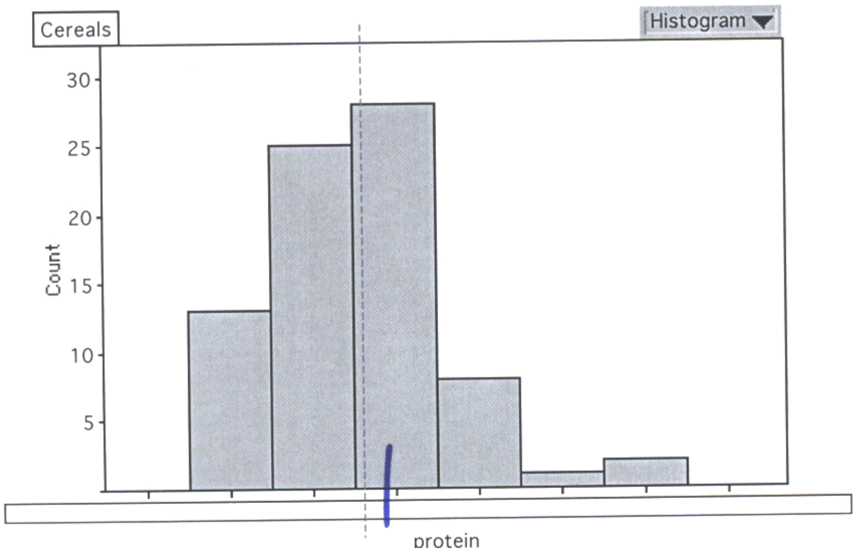
21. I: OK. That's pretty straightforward—

22. P: It's the definition

23. I: Yeah, the definition. All right, now if you were actually, let me ask you to do this. Would you place a line where you think the mean might be; the arithmetic mean.

24. P: Well, it's probably to the right of that somewhere [picks up pen and places blue dash, on figure 2].

HANDOUT 1



25. I: OK. Now why did you say that it would be probably to the right?

26. P: Well, you've got this, this right tail [runs pen over three bars on right side]. It looks like it's stretched out a little bit. The right tail—any outliers will pull the mean up toward it.

27. I: Why does it pull the mean that direction? When you say "it pulls the mean up"—

28. P: It um—if you have a large value in your sample, say on the right side, it can dominate all the rest of the data points. So that in effect you can control where you want the mean of a set of values to be by simply moving one of those values way out. The size of it in the average dominates it.

29. I: OK. What is it about the average that makes it so that it is dominated by that one data point?

30. P: Because the numerator is a sum.

31. I: OK. All right, now, I had told you earlier that there were 77 boxes of cereal there. Would know—would there be any difference if I had more than 77 or less than 77, to where you would place that mean?

32. P: No.

33. I: OK. The count isn't going to create a difference?

34. P: Um, well—let's think about that for a second.

35. I: OK

36. P: If you have a, if you have a similar shaped histogram [points to given histogram – figure 2], let's see, um, no, I don't think it would make any difference.

37. I: OK. Well, why—you started to say that there's similar shaped histogram, then you changed your mind to going that direction and just said "I don't think it's going to make a difference". What made you change your mind there?

38. P: It looks, if the histogram is of this shape, um, everything should be proportional. So if you increase the sample size, but kept these bars like this [circles bars of histogram with pen cap], then in effect you would have—say you double the sample size in such a way that you have two of each of these, then, for example, algebraically you could factor a two out to the left and there'd be two 2n in the bottom and you could cancel the twos, if you want to see it that way. But it basically what it saying is the distribution remains the same, so it doesn't really matter how big the population is, what it says it is.

39. I: All right. What if—let's say that these numbers are one, two, three, four, five and six [points to each bar in succession]. Would this mean [points to mean P placed on histogram] be in a different place if this were, say, one hundred, one hundred one, one hundred two [points to each bar again], and it was transformed up to that

40. P: No.

41. I: There wouldn't be a difference at all? Why is that?

42. P: All you're doing really is adding 100 to each of the observations and once again you could isolate—you could see it algebraically by just pushing the 100 out. You would have, what did you say, 77 times a hundred. And so you would just be, you would just—in other words the sample mean is invariant to adding a number to the observations.

43. I: OK. [Time 6:11] All right, this next question I'm going to ask you— Actually, I'm going to give you a blank piece of paper, which is always a scary thing [hands P a blank sheet of paper], but—The question is this, if there were no such thing as the arithmetic mean; it had never been written up, would you invent it?

44. P: Sure. I think because it's an average and people are always interested in how you might declare a typical value of a data set. This would be one very logical or reasonable way to give you a typical value if you give me a set of numbers. If you want me to somehow summarize a set of numbers by giving you a typical number then the mean would be sort of a natural way to do it.

45. I: So, if I said that the median was something that we had and mode was something we already had, but mean didn't exist, would you invent the arithmetic mean for another reason other than just typical value. I could use median as a typical value as well

46. P: Absolutely. In fact, I would say it's a much better typical value. But, uh, because I think people understand what a value means if you, if you split the sample into two equal parts. Now, if you have a symmetric data set then I think people

understand what the mean is. But I don’t think people have a good feeling for the mean if the data is not symmetric or roughly symmetric.

47. I: What is it that the arithmetic mean gives us—that, that it was invented?

48. P: Um. Now I don’t know historically why the mean was invented. There are certain places that it arises in statistical analysis that makes it natural because if you parameterize the distribution and one of the parameters is the mean of that distribution, the expectation, then the mean, the sample mean is the natural estimate of it.

49. I: OK

50. P: Um, it probably comes from the normal distribution—all the way back to the 1700’s when Laplace did all that work. It’s mathematic—when you get to the normal distribution and all the things that flow from the normal distribution; least squares and so forth, it’s a natural, it’s a mathematical natural thing—

51. I: What do you think about the mean makes it so natural with the normal distribution? I mean, it does flow naturally, but what is it that makes it flow so naturally?

52. P: Well, if you are willing to accept a bell-shaped, normal curve, then I can characterize the curve with two values; the mean and the standard deviation. So, all you have to give me is two numbers and I can give you the specific normal curve you are interested in. So it characterizes, those two numbers characterize the normal curve.

53. I: An interesting thing I run into in statistics or I have run into in statistics before is when I don’t have a normal curve and yet we still try to describe it with the mean and the standard deviation. Why do you think that is?

54. P: Because most people still imagine, for example their trying to model something, they often have a symmetric population. So the mean, the median are typically very close, it doesn’t much matter. And if people aren’t careful and they are willing to risk data contamination and those effects, then they’ll use the mean.

55. I: Yeah, it’s interesting.

56. P: It’s also much, it’s more, by the way, it’s much easier to compute the mean than the median in some kind of relative sense. If you’re computing the median of ten billion numbers, then are you going to order them all? Are you going to write a computer program to order all ten billion numbers? It’s quicker to compute the mean.

57. I: It’s convenient.

58. P: Yeah.

⋮

69. I: It's usually introduced in the middle school. [time 11:17] Um, here's a fun question for you. I'd like to ask you, without using the algorithm of sum everything up and divide by how many you have, how would you define arithmetic mean? And could you write that down once you, sort of, come up with something you're happy with?

70. P: You mean you're not going to let me sum up the numbers and divide it by n?

71. I: Right. I'd like to see if you can come up with another way of describing the arithmetic mean.

72. P: Well, we could, uh; you could put the data points. Suppose you have three data points [draws line, figure 3]. One there, one there, and one there. And you put a pound of clay there and a pound of clay there, and a pound there [points to each hash mark in succession]. And you find a point [draws triangle under the line, figure 3] at which it balances; that will be the mean.

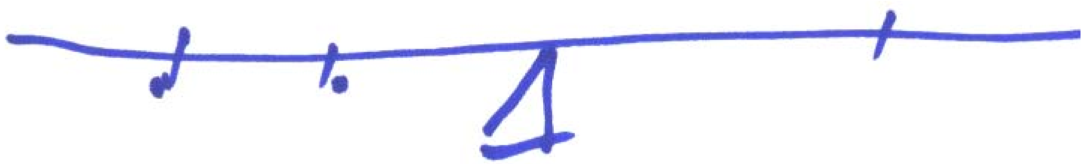


Figure 6

73. I: OK

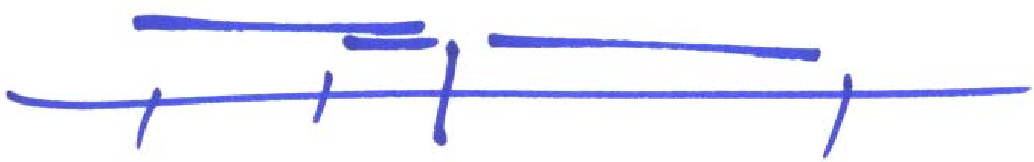


Figure 7

74. P: So, and then in some sense, it's a centroid of the data set; it's a centering value to sort of balance the data. Um, if you said I've got three data points [draws second line and marks three hash marks, figure 4] and I want to pick a number which is

closest to all the other, to all the data. Then you might start, you might take a test point here [adds fourth, longer hash mark between second and third marks] and that distance there, and that distance and that distance [makes horizontal lines above for each distance] and then you're going to, again, you're going to use the standard formula for distance; the square root of the sum of the squares; and find the one that minimizes. So in that sense, the mean would be what would solve that problem. And the point is the closest to the other three, in essence. So in that way, in a way it is that point which best represents the data in the sense that it's closest, in some overall sense to the data; to the data points.

75. I: So, that minimum point of distance when you take all of them. Ok. All right.

Figure 8

76. P: Um, you can also, you can, I suppose, I have to think about this a bit more. But if you put weights on data [writes "W x" on paper, figure 5], and then add them up, my guess is that if you put equal weights of one over n on—[writes $\frac{1}{n}$]assign those then; you could argue that — of course, that's the expectation of the empirical distribution. If you want to talk about empirical distributions. In the sense that—actually, that to me is the easiest and the simplest way to introduce it in, say, mathematical statistics courses, a first course; where students know what expectation is and what population is, are. But the thing that's interesting is that, even more advanced students somehow don't make the connection between the empirical distribution and its expectation. All of the theory of expectation and populations and distribution that they've learned, they don't somehow make that connection.

77. I: So you would actually introduce it this way, then.

78. P: It's one way, you could say, you've got a sample. I'll just write it this way [writes x1,—, xn]. And you put a weight of one over n [writes $\frac{1}{n}$ under each x]. So now you have a population or a set of values and probabilities on them. So now you have a distribution. And you could say, "suppose you have a random variable that takes these possibilities with these probabilities, what's the expected value?" It's just the sample mean. And the and the variance of this distribution is the one over n version of the sample variance.

⋮

85. I: That's interesting. [Time 16:40] I'm going to ask you to do something that's not quite so abstract. Ok? This is a fun one. A class of students—this is a problem I got from a—I've actually changed it a couple of times to make it a bit more useful [places problem in front of P, figure 6]. These, basically, you've got a class and their average grade on the test, the arithmetic mean was 68. The average grade of the students who passed was 80 and the average grade of the students who failed was 64. So I am asking you if you can tell me what percentage of the class passed.

> A class of students took a test. The class average on the test was 68. The average grade of the students who passed was 80 and the average grade of the students who failed was 64. What percentage of the class passed?

Figure 9

86. P: Ah [laughs and moans a bit]

87. I: And you may take your time in solving this.

88. P: So, let's see what that looks like [draws horizontal line and places averages on, figure 7]. The average is 68, and you say the average of the students, the average of the students who passed was 80. So, up here, there is 80.

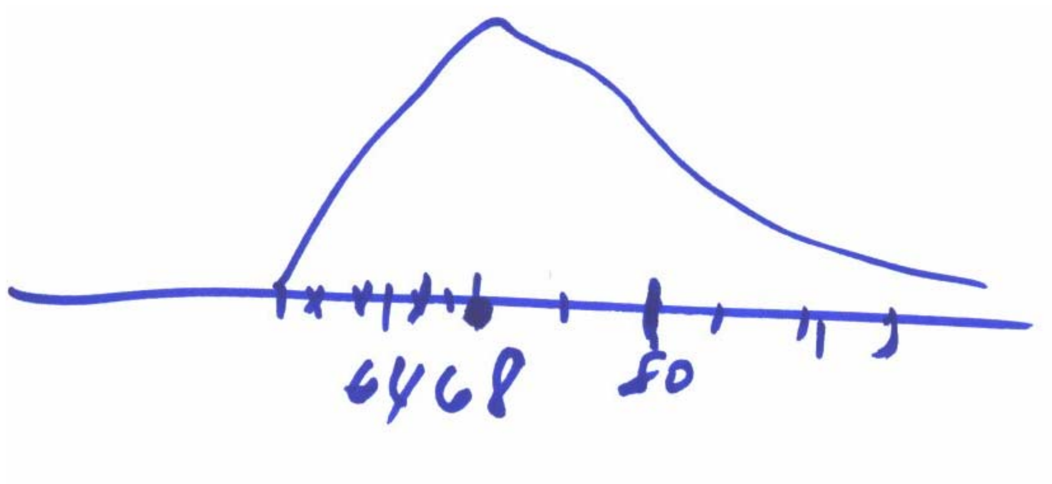


Figure 10

89. I: Oh, I'm going to put one more stipulation on this once you write that down, go ahead.

90. P: Those who failed, the average was 64.

91. I: Right, now here's the additional stipulation there. I don't want you to use algebra.

92. P: OK.

93. I: So, do not use algebra to solve the problem.

94. P: What do you want me to know, the percentage that passed? So you're telling me that I can actually solve this problem?

95. I: Mm hmm.

96. P: That's interesting. [16 second pause]

97. I: I'd love to know what's going through your mind as you're looking at it. What are you thinking about?

98. P: Well, I'm just thinking about what the distribution must look like for this; for these students. Um, it strikes me, without, I'd have to think about it more, but it just strikes me that it looks something like that [adds curve drawn above line, figure 6].

99. I: OK.

100. P: Because the—the class average is 68—umm, there's a lot going on here, possibly. I'm probably making it much more difficult than I should, but, but you have 80 up here [places pen on 80 hash mark]. Um, so there's a bunch of scores up in here

[adds unmarked hash marks above 68]. I'm not sure what, what's—do I even know what's passing?

101.I: No, I'm not sure—I can give you a passing, but I'm not sure it is going to help you.

102.P: [talking over I] —because 64—

103.I: —it might muddy matters.

104.P: OK. A 64, the average down here. It seems that they would have to be bunched up a bit more here to get the average something like that [places unmarked hash marks between hash marks labeled 64 and 68]. So, um, you want the percentage of the class that passed [23 second pause]. Hmm, I would just guess. I don't know.

105.I: OK. Um, earlier you had said that one way to think about the arithmetic mean is to think about it as a centroid or a point of balance [points to earlier drawings for definition, figure 3]. Is there something about a point of balance that might help you figure out a percentage? Because these are each points of balances; one is the point of balance for the passing, one for the failing— Is there anything about that might help you, without using algebra—to solve this?

106.P: [22 second pause] No, I don't see—again it seems like the [pauses 2 seconds and begins to mumble] let's see the percentage—the average grade—hmm. I don't see—I don't see how that's going to [6 second pause]

107.I: OK. All right, I might come back to it.

108.P: OK.

109.I: [Time 21:28] But that's OK, we'll put it over there. All right, now, this one is a little more straightforward that that one. This one's much more straightforward, actually [places bar graph, figure seven without writing on it, in front of P]. We're going to pretend here that this is depicting building heights. All right? And there are seven buildings here [runs finger over bars] and I would like you to show me where you think the average building height is and how you determine that from this picture.

110.P: Mmm. [11 second pause]. Again, I would probably put it right around here, about here [places arrow pointing up as shown, figure 8].

111.I: Around building four?

112.P: Yeah.

113.I: That height [points to top of bar labeled 4]? So if I were to—Let me just do this so I can see it [takes green pen and draws dotted line even with building four]. You're saying that it's about right [makes noise], I didn't do it real straight, but you get the idea, right there? OK.

114.P: Yeah.

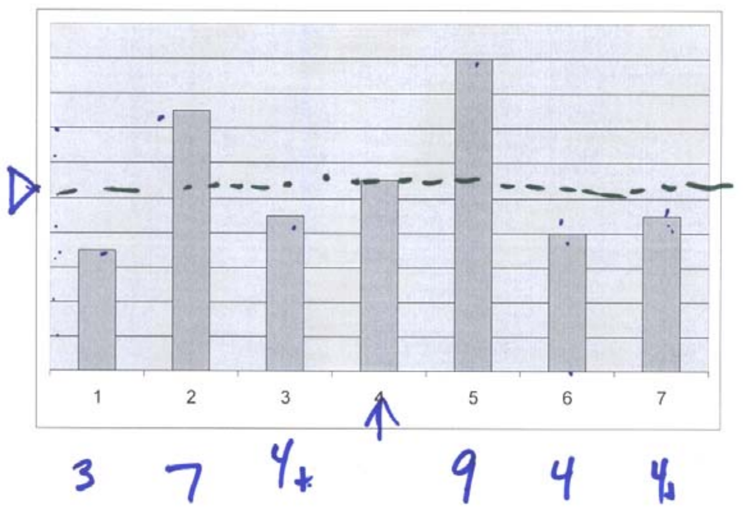


Figure 11

115.I: Now, how did you determine that?

116.P: Well, there's more here [runs pen over building 5]. On the other hand, this is out further [points to building 2], so if I was going to, if I was going to make adjustments, I would probably adjust it in this direction [points to right of mark shown for mean]. Although, that's the center, you see. The fact that this is big [points to building 5], but closer to the middle and this is still big, but still further out [points to building 2].

117.I: I'm not sure I'm following what you are saying.

118.P: Suppose this was [points to building 5], I don't know, put weight on it again. I'm sort of ignoring these two [points to building 7 and 3]. Supposing these are roughly the same, OK?

119.I: OK.

120.P: And, so, if I was going to try to balance this system, then this big one [building 5] is going to tend to pull it down this way [puts pen to the right of building 4], but, here's another [points to building 3] that's not quite so big—

121.I: Oh, you're balancing it this way [runs hand across chart from right to left]

122.P: Yeah.

123.I: I'm thinking this way [runs hand top to bottom of the chart].

124.P: Oh, I see.

125.I: I'm thinking, here's building one [points to first bar], here's building two [points to second bar]—I'm sorry, I was trying to understand you. This is building three [points to third bar] and, OK, this is a short building [points to building three] and this is a tall building [points to building 5] where would the average height be, about? Are we still—?

126.P: So, this is heavier

127.I: Right.

128.P: I mean, you want me to put an average, along this axis [runs pen along horizontal axis].

129.I: No, I want an average height [places side of hand along horizontal line drawn and slowly moves up and down] along about this high—that's why I put this line—

130.P: Oh, oh. [speaking over I] I'm sorry

131.I: No, that's OK.

132.P: I see.

133.I: That's why I was making sure we were communicating there.

134.P: All right. Sorry, yeah, so you want the average height of these things

135.I: Yeah, right.

136.P: well, let's see—

137.I: I do see what you were doing—

138.P: Ok, that looks about as good

139.I: Now, how did you determine it? Once you said "that looks about as good", how did you determine it?

140.P: Oh, these four, these four right here [points to bars one, three, six, and seven] are under. These two are over [points to bars two and five], but their over by a bit more, so the "bit more" [holds pen horizontally over green line] will off-balance these four [one, three, six, and seven].

141.I: OK, so—

142.P: And they're small—

143.I: So in your mind—

144.P: —it's hard to, it's hard [turns paper 90 degrees and looks]

145.I: Yeah. Yeah, actually, that's why I put the horizontal lines there; to try and help you. I did it the first time without the horizontal lines that was tough. Um, but let me make sure I understand what you're doing. You're saying that this piece right here [runs finger over part of bar 5 that is over green line] is going to off-balance with these little pieces here [runs finger over space between bar six's top and the green line].

146.P: Right

147.I: So, what you're trying to do, in a sense, is trying to get them all as if they were the same [runs hand across the green line]?

148.P: Yeah.

149.I: OK. All right, now, earlier, again, you had said that if, mean is like a point of balance; which is what you were working with here [points to definition, figure 3]. But if mean is like a point of balance, how is this average height of the bee—of these buildings like a point of balance?

150.P: Well, I guess you could think of it this way [places triangle to left side of bar graph in line with green dotted line, figure 8].

151.I: [laughs] OK. And so—I have, I have a problem with, if we think about it that way. And that is, if I tip this [in]this direction, like this [turns paper 90 degrees]. I'm going to—I'm going to make the statement that I think that this side of the line [points to bars below green line, or to right] is much heavier than this side of the line [points to bars above the green line, or to left. time 25:24].

152.P: Um—do I believe that? No, I'm not sure I do. Um, you're going to average these, these numbers [points to spaces above bars one, three, six, and seven], right?

153.I: mm hmm. Actually, when I look at it the bars are heavier. But you're looking at something different, aren't you?

154.P: I may be.

155.I: Because if I add up this bar, that bar, this bar, that bar [spans finger over bar from bottom of chart to the green line for bars one, two, three, four]—this side is definitely heavier.

156.P: mm hmm

157.I: But I think you were looking at something else when you pointed to these four [points to bars one, three, six, and seven]—were you looking at something else?

158.P: I was probably thinking of the height of these things [runs pen from top to bottom of bar six and seven]. I mean, I guess we could start counting, couldn't we?

159.I: [laughs] we could.

160.P: That's about three, four, [counts bars one and three] and what's this? Seven, nine [counts bars two and five]—so we have sixteen here. What do we have here [waves pen over six and seven]? I'm going to leave this one off [points to bar four, mumbles] that doesn't look too bad.

161.I: OK, now you said "I'm going to leave this one off" [points to bar four] why did you do—

162.P: Well, that's right at the mean.

163.I: OK, so, ah, let's say that—you said that this one's about three and you said this one's about four, this was about seven, this was about nine, that one's four—or just a bit more than four, so four plus there and this one's also four plus, just a little bit more than four [labels each bar with number, as shown figure 8]?

164.P: mm hmm

165.I: All right and you said—as you looked at it you said, "Yeah, that's about right" um, what were you doing?

166.P: Well, I was probably looking at the four and four and four and four [points to numbers under bars seven, six, three and one], and that's about 16 and that's about 16 [points to bars five and two].

167.I: OK. And you're saying since they're about the same, that's going to be like this point of balance here [points to definition, figure 3]?

168.P: mm

169.I: OK. Thank you. Making sure I understand. [time: 27:27] I have another thing that was developed by Fathom [places dot plot in front of P, figure 10]. Um, this is, uh, comes from a set of data that comes with the package and it's a dot plot. And any where you see the dark [points to dots that are colored], it means that there's more than one data point.

170.P: Oh, ok

171.I: OK? And so you can't see the full picture here. In fact, if I were to try and show you the full picture I'd have to show it to you this way [turns paper 90 degrees]—

172.P: Yeah.

173.I: And, um, let me just put a mark on here. The mean of this data set is right there [places dark hash below 10th line of dots]. It's on that part of the dot plot.

174.P: OK

175.I: Um, if I were to, uh, add a data point to this, this whole set, and I were to add it right here [places red x below set of dots two rows to right of mean, figure 10]; on that line right there, in that set of dot plots, how would that change the mean?

176.P: It would move to the right.

177.I: OK, Why?

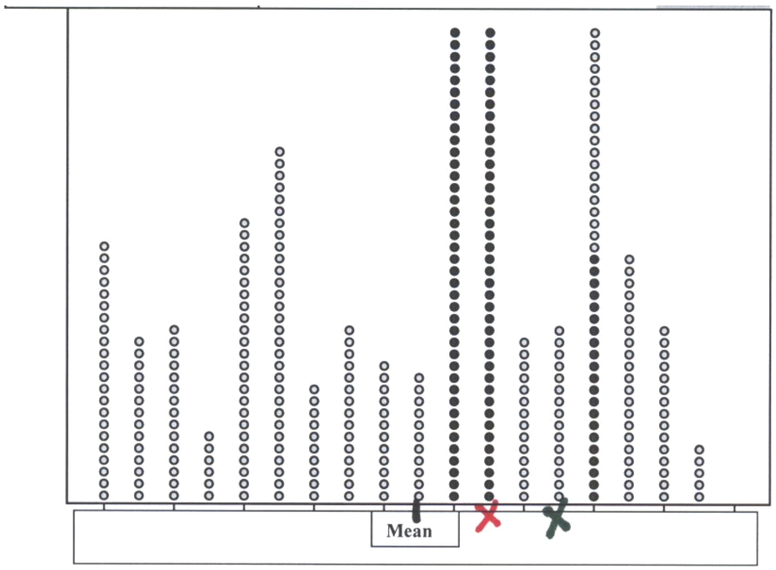


Figure 12

178.P: Um, because if you add a number that is above the mean it's going to move the mean in that direction.

179.I: Why— this is the thing that I'm still trying to make sure that we— I understand; and that is when something is put to the right, what about putting it to the right makes the mean go up? Go towards the right?

180.P: Well, you're, you're going to add a value which is bigger than the mean to that set of data and then you're dividing by n plus 1 or whatever and so it's going to increase the mean by, uh, I don't know what it's going to increase it by

181.I: OK. Given this set of data, is there any way you could know—given particularly that we don't know how many are there—is there anyway you can know how much the mean will go up? By placing it in this [runs finger over column of dots where extra dot was added]—

182.P: At that value?

183.I: At that value. I almost said bin, but that wasn't right.

184.P: Yeah, I think you could. Um, if you add a value to the right of the mean, so the new value of the mean is going to be the old mean times the old number of them plus the new value, divided by new number of them, right? So, you can clearly separate the pieces from the old part. That is, the old x-bar, the old mean and the piece that's added to it to get the new one.

185.I: OK. The one thing that I haven’t given you, though, is the old number of data points. So that limits what you’re capable of doing because you don’t have that.

186.P: That’s true. I have to have that.

187.I: Now, let’s say that instead of putting it there, I put it up here [places green x four columns to right of mean column, figure 10], four sets over.

188.P: OK

189.I: Now, what would happen to the mean, first of all?

190.P: It would go up.

191.I: OK, would it go up more or less than before?

192.P: It would go up more.

193.I: Do you have any idea how much more?

194.P: Um, yeah, I think you might be able to find out the difference between these two and divide by n or n plus 1 or something.

195.I: OK, so um, the difference between these two [spans thumb and index finger between two “x’s”] is two. We’ll say that these are going up by units of one. So the difference between the two, are two.

196.P: OK. So you, basically, you have the old sum, plus two over n+1, don’t you?

197.I: Right.

198.P: And you want, so you want the change in — the new mean to the newer mean

199.I: Right, exactly—the red mean to the green mean, will call it.

200.P: Um, So uh, do you want me to do the algebra?

201.I: Are you doing it in your head? I think that—

202.P: Well, I’m working through—

203.I: —yeah, go ahead and do a little bit of it. See if you can figure it out.

204.P: So you’ve got, you’ve got uh n x-bar which is the sum, plus a, plus b— I’m sorry, we’ve got— that’s not it at all. We’ve got to—n x-bar plus a and we’ve got n x-bar plus b, both of them divided by n plus one; take the difference. So, the n x-bars go away and the difference is simply b minus a over n + 1

205.I: OK. And that’s then how much the mean will then change. So, b-a is 2, so it’s going to be 2 over n minus—

206.P: n plus 1

207.I: Yeah, n plus 1.

208.P: Right?

209.I: Right. OK.

210.P: Did that answer—?

211.I: Yeah. Sounds good. Again, this one [points to dot plot] seems to look more like, like your drawing [points to figure 3] of arithmetic mean than this one did [points to bar graph]. And yet we're still are using pretty much the same concept when we're doing that. Um, one of the differences I saw, and I just want to see if there is a way to word a question in here, so hang in there with me.

212.P: OK, sure.

213.I: Um, one of the differences I see here is, here [points to bar graph] you came up with matching sums; you said that these four numbers here [points to bars one, three, six and seven], pretty much summed up to that [points to bars two and five], to those two numbers there. And so you were talking about matching sums. Here, you're just talking [moves hand over dot plot] about moving this black mean bar slightly up based upon how many data points there are and so you're not matching anything. Is there some way that you can relate what you were doing here [points to bar graph] with this understanding of balance? Um, this, this wasn't really finding a point of balance. Well, at least it didn't look like finding a point of balance.

214.P: Right.

215.I: Is there any way you can relate that?

216.P: Mm, I don't see it off hand.

217.I: OK, I'll give you a little time—

218.P: —on the other hand, I'm not real clear about this problem anyway.

219.I: Really?

220.P: Yeah, I'm not sure what you were asking about it, so—

221.I: What, um—

222.P: Because I was thinking of this one [points to bar graph] in terms of if it were a histogram and these were masses [points to each bar].

223.I: Right, right

224.P: And where you wanted to the mean [places hand above blue mark originally made to show mean] —

225.I: I see

226.P: You see? Yeah, and I think you weren't thinking—

227.I: Yeah, I need to make it more like a building than a histogram. I mean

228.P: Yeah, perhaps though —.

229.I: Make it look more like a building when we—yeah, and when, maybe when I originally did this one I had this little story with it. Maybe the story actually would have helped you—the story was you were looking at a building site and—

and were seeing these buildings and the builder says “I know what the average height of all those buildings are” and you try to guess about where it is. Maybe that would have—

230.P: I’m so conditioned to thinking this way [points to dot plot], that it may be this [circles bar graph]—it’s still probably hard for me to think—I understand what you are saying now. You wanted the heights here and the average height. But the minute I start thinking that way, I still start seeing a histogram and I start—

231.I: That’s because I made it on Excel with their little bar graph tool so I could make it neat.

232.P: That’s what I said, but it’s an interesting conceptual issue, really, isn’t it? Because my thinking about this [points to dot plot] interferes with my ability to sort of work out the heights –

C.1.3 Partial Transcript of Task Based Interview with L

11. I: Get some paper, yeah. All right. The first thing I’m going to give you is; this is a histogram of- I just made it up. It’s boxes of cereal and this is how many boxes of cereal [runs finger along the vertical axis, figure 1. Note numbers along horizontal axis were added later]. The counts on this side. And down here [puts finger on horizontal axis] are the number of protein grams, grams of protein in the box of cereal. So like this bar [points to first bar on the left] is 11 or 12. Probably 12 boxes at that count. So like it could be three, it could be five, it could be—it doesn’t matter at this point what it is but there is a number of grams of protein. So this is just a display of 77 boxes, all together. Okay? So, if we were to count up how many [points to each bar in the histogram] there were all together, it should be 77.

12. P: So, here means, for all this thirteen boxes each has that gram of protein?

13. I: Yeah, they’re going to have so many grams of protein.

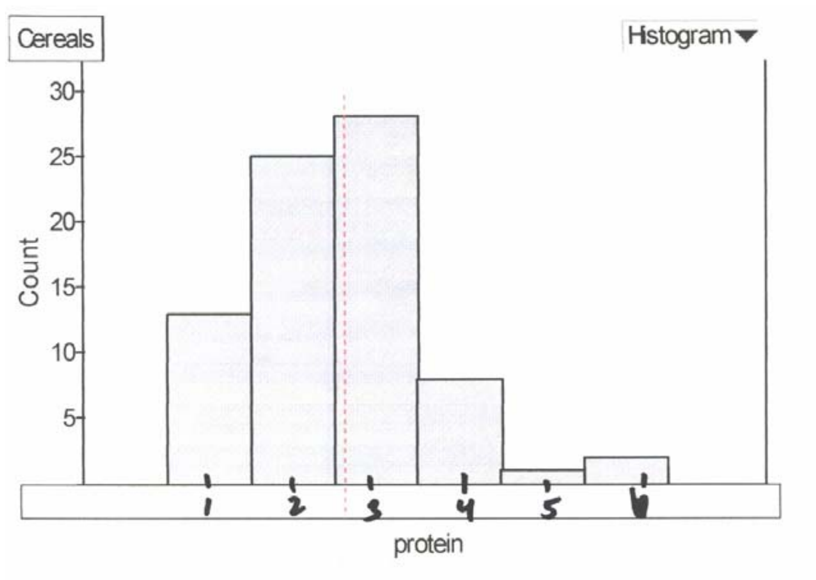


Figure 1

14. P: OK, uh huh.

15. I: And this is one unit [points to first bar] . . . it's a unit and another unit [points to second bar] and it is uniform in between.

16. P: I see.

17. I: Okay?

18. P: Uh-huh

19. I: Now, what I did was, I put a red dotted line there [points to line on histogram]. Can you see that, the little dashed line?

20. P: Yeah, I see that.

21. I: What that does is it takes the area on this half and the area on that half [uses fingers to span bars on left side, then right side] and it divides it sort of equally. I tried to get it close.

22. P: So that's the mean

23. I: Yeah, Okay. So, what did you say that was?

24. P: That's the mean.

25. I: You think that's the mean, there?

26. P: Yeah, if the area is the same, there would be the— if I get this thing right— okay, that the —this represents the protein, the number of protein, the part of protein, for each individual box that are in this category [has histogram in hand and appears to be running little finger up one bar of the histogram]. There are thirteen

of them with that much protein; you've got 25 of them with that much protein [points to next bar over]. So, I would tend to think that you would multiply the number of boxes with the protein level; add all of them up divide it by the number of boxes, would be the average value of the protein for all the boxes. Now, the area is relatively, for this different part of boxes, would be, it could be—let me see—let me try to figure it out—the mean, the true mean value.

27. I: OK, sure—

28. P: It would simply be the area, the area—okay, the area does not seem to represent the total—each column, just multiply the number of boxes, by that particular interval [spans finger across the width of one bar] which is not the total protein level. And therefore, when you add all the area together, it is not the total protein for all the boxes. And therefore, dividing this in half [runs finger along the red line] does not truly represent the mean value.

29. I: OK. So it's not exactly the mean, but it's

30. P: Yeah, it's not really the mean. That's how I look at it.

31. I: OK. —No that's good [both laughing a bit] that actually was very helpful. Okay, um, are you familiar with the median?—number

32. P: Uh-huh [positive]

33. I: Where do you think the median would be the median number of boxes of cereal? Or gram, protein grams would be?

34. P: Yeah, uh, so again the focus is not the number of boxes; the focus is the protein, right?

35. I: Right.

36. P: The median for all the boxes together?

37. I: Uh-huh

38. P: Yeah, so the way I would determine that would be multiply the number of boxes in each column by the protein level, which is actually the area—not represented by the individual column, but the area extending that all the way to the, to zero.

39. I: Okay—

40. P: So this area [runs finger along first bar], plus the next column, it would be all the way down here [runs finger from top of the bar over to the vertical axis]. So, you actually have to add up all this area [runs finger over all of the bars] and then divide it by the number of boxes. So, then from this diagram it would be difficult to just, um, tell where is the median.

41. I: Okay, that's good, because I actually made the diagram difficult on purpose.

42. P: But, I tend to think that a, just a rough estimate, that more weight would be given to those that, the column that's larger, the larger box of numbers. And I would say that based on this, the two largest number [points to second and third bar] and

looking at this one [points to first bar] and, but this one [points to fourth bar] is much higher protein level. So, even though it only has nine or eight there, the boxes, it would out weight this one, this first column [points to first box] there from the thirteen boxes. The protein level is considerably lower.

43. I: Okay, so you're saying that when you have a higher protein count, you can have a shorter—

44. P: Shorter, or smaller box

45. I: But it still balances—

46. P: It still balances the other one [points to first bar]. But if I really have to take a quick guess, I would the red line the areas probably—the median—the mean value would be somewhere close to the red line, or even to the right of the red line.

47. I: Okay, so you're are saying—

48. P: The one with the higher level [lays index finger pointing to the right on the histogram] would out—would carry more weight; when you do the average process.

49. I: Okay, so what your saying is this red line, maybe to the right of it because these boxes here [points to three bars on far right] carry more weight.

50. P: Carry more, a higher protein level.

51. I: So, a higher protein level will carry more weight.

52. P: Yeah, so indeed this red line represents the true average of the individual columns, then the mean value of the protein for all the boxes together should be to the right. Those with high level, carry more weight.

53. I: So, there are two things that you're looking at. You are looking at the height of the bar [points to tallest bar] but you are also looking at how far [runs finger to right] out it is.

54. P: Yep, the protein level.

55. I: All right. Now [time 7:32] I said that there were 77 boxes here. What if this count weren't shown and I said there were 158 boxes, would that change your answer at all? In other words, you've got this same picture —

56. P: Yeah, it would—in the center it wouldn't—you can stretch the scale; you can go from zero to thirty [puts fingers on vertical scale] or you can go from zero to three hundred. But it will not change the mean value.

57. I: OK. Where it's placed?

58. P: [Nods head] Where it's placed.

59. I: Because these didn't change? [points to protein values along horizontal axis]

60. P: Yep, because the protein level didn't change. And the vertical distances didn't change [runs finger up one bar].

61. I: Okay, all right and—and—

62. P: The one way to get averages is divide in each column the number of boxes by the total number and that fraction will not change if the shape of the graph remain the same.

63. I: Okay. What if—like, let's say that [picks up pen and writes numbers across horizontal axis, figure 1] I had labeled these as 1 gram, 2 gram, 3 gram—these are low gram protein cereals. And, what if I said to you, Okay, so you're saying the mean is like 2. 6 maybe 2.7. Someplace up in here you were saying [pointing in third bar], maybe even at 3.

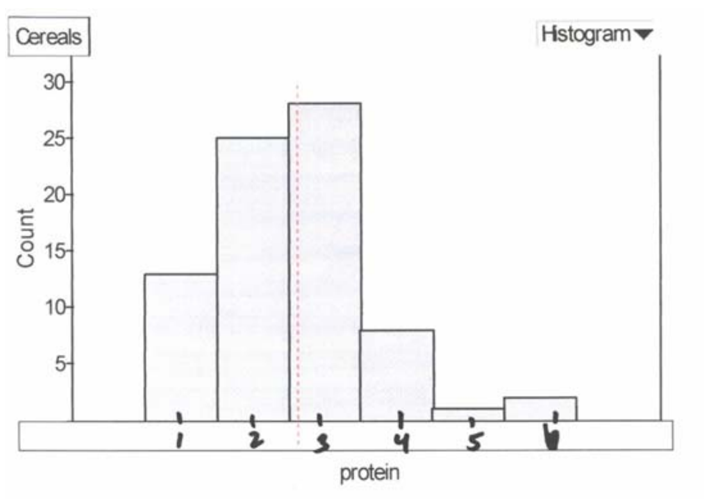


Figure 1

64. P: If this is, you're talking about the, uh, the red line, right?

65. I: Yeah

66. P: The red line is like 2.6?

67. I: Yeah

68. P: But again, that is not the true mean of the protein level for all the boxes. And it would tend, I would tend to think it would be closer, it would be larger than 2.6 and it could be three or even larger.

69. I: Okay. Now, if I made this 100 [points to 1 under bar one, then to each successive bar], 200, 300, 400, 500, 600—would that change where you put the mean? Not the number, I mean the number would obviously change but would it change where you put it on there? [Time 9:24]

70. P: If the scale are even—

71. I: Yeah, the scales are even.

72. P: If the scales are even then you won't change the relative position.

73. I: Okay, so it remains invariant to all that?

74. P: The vertical position remains the same, but the absolute value would change.

75. I: All right, yeah, true!

76. P: They all pass—[laughs]

77. I: All right, thank you.

78. P: Very interesting the set diagram and the questions.

79. I: We're going to get even more interesting.

80. P: It's more and more difficult, right?

81. I: Let me ask you this question [Time 9:57] Suppose the arithmetic mean had never been invented or discovered, whatever, it wasn't there.

82. P: Yeah, yeah, OK.

83. I: Do you think that in the things that you do that you might need to invent arithmetic mean?

84. P: Uh, yeah, sure —

85. I: Ok. Why would you need to invent it?

86. P: Um, I guess it's the it really is just to make life easier. Once you know the mean value and just multiply it by the total number, you get the total value. Rather than having to add the individual, if there is very large variation. First of all, if there's no variation, there's no need to get the mean value and if there is variation, then it makes life easier or makes it simpler to just provide the true average.

87. I: OK. Is there some aspect of the mean that you say, when I think about using arithmetic mean its better than using median? Uh, median—you made the statement that you can multiply and figure out how many total you have —

88. P: Well, let's first clarify the definition of how you define the two, the two—what are the definitions of the two in your mind?

89. I: OK. In my mind, arithmetic mean is something similar to what you've been saying already, which is sum everything up and divide by how many you have. And median is taking the data set and splitting it in half and saying, what's that half way point? Where is the middle of the data—given an ordered set of data?

90. P: Yes, yep, yeah. OK

91. I: All right? So, is there something about mean that you say this is definitely going to be more helpful?

92. P: Yeah, well it is a very good question depending on the focus, on the point of application.

93. I: Right. OK [laughs]

94. P: If uh—let me try to answer your question in this way; by providing an example. Let's say we are from the male and female and children and different ages in the auditorium. OK, now we try to come up with the—we try to ask the question that the uh—let me see what is the best way—what is the average age, OK? Of all the audience in the auditorium? And that would require to get the true, the mean value. On the other hand, if we just say that the—let's say that you have about 10 above 60 and then there are 20 of them between 60 and 40 and then hundred would be younger than 40 or something. And you want to get the true average or you can apply it to life expectancy as well. So the number of people that would live. So the number of people that are exactly 45 in the auditorium, and the number of people exactly 35 years in the auditorium; you add them up, divide it by the total number of people, you get the arithmetic mean. But if the—so if the focus is what is the mean, average value of the age of the people in the auditorium, that would be the one you should use. But if the focus, if I want to know who, the problem is how many people would be above a certain age, and I want to determine what is that median age, then just simply—whoever is—let's assume the median is 42. Whoever is younger than 42 go to one, left hand side of the room; older go to the right hand side and then just split them into two. Those older could be only one years older or they could be 40 years older, but I don't care. You only count one for each.

95. I: Yeah, OK.

96. P: And that would be the median. That would be the—in that case that would be more meaningful to you to use.

97. I: Right. So its context?

98. P: It's context in that case.

99. I: OK. Good. All right, you've been teaching here at Penn State for a little while [laughs, TIME 14:12]

100. P: [laughs] OK. Quite a while.

101. I: And you work with engineering students who I am assuming they have some math background as they come into this—

102. P: Oh yes, yeah.

103. I: So my question to—

104. P: Good math background.

105. I: Good. Have you, as you've taught, do you ever run into any problems with your students' understanding of arithmetic mean or do you think they come in pretty solid in their understanding?

106.P: I guess they, uh, again it depends on the context. In most cases, the statement of the problem or the definition is pretty clear cut that when we talk about a mean or average everyone tends to refer to the same quantity.

107.I: Yes.

108.P: But the going back to your, using your previous question as a sample, that there could be two average, depending on what is your focus and either one of them could be the meaningful average to use.

109.I: OK. And sometimes the students don't always—

110.P: Don't always

111.I: They seem to go to one but not the other

112.P: Yes, to the one actually.

113.I: Um, I'm going to ask you to try and define arithmetic mean without using "add everything up and divide by how many you have". [Time 15:15] Is there another way you can think of defining arithmetic mean.

114.P: Hmm [4 second pause]. Yeah, I would just define it as the true average for all the items, accounts for all the items involved.

115.I: Ok, so it accounts for all the items involved.

116.P: Let me just give you a simple example to clarify the difference. Let's say there are 11, 10 persons in the room and each, in their pocket, have some money. Okay, if we want to, if the question is that, what is the average pocket money that they have for all this ten persons, then we just get all the money together and divide it by 10. Okay?

117.I: Okay.

118.P: That would be the true average. But let's assume nine of them have less than ten dollars in their pocket and one person, they have several thousand, say ten thousand. But that ten thousand out weighs everyone. But if the focus is, say, who—what is the mean value of the people— among those 10 persons, the median of the amount of the money that half of them would carry less and half would carry more. In that case, the person who has ten thousand would not out weigh the rest.

119.I: Right. So you're saying that one thing about mean is it takes into account when you've got something unusual.

120.P: You've got something unusual, that's very true, yep.

121.I: Like ten thousand instead of the one dollar.

122.P: So, when you have 10 persons, you have one carry much more money than the others, and you divide it by ten, that would be mostly that one person's money, you get an answer very close to that. But, in reality, the focus is not the true average then that would not be meaningful to use. The majority of the people don't have anything close to that.

123.I: Yeah. OK. I have a task [Time 17:36] for you to try [places class average problem on table, figure 2]. Ok, here's a problem for you to try. If you could read that to yourself and if you have any questions, ask me and then I'm going to ask you to try and solve that.

124.P: [pause to read and speaks to self as reading] Ok, that's the class average and the class was 68, "the average grade of the student who passed was 80 and the average grade of the student who failed was 64." Mm-hmm "What percentage of class passed?" Oh, very good questions. I'm going to try to—

A class of students took a test. The class average on the test was 68. The average grade of the students who passed was 80 and the average grade of the students who failed was 64. What percentage of the class passed?

Figure 2

125.I: All right, now, let me put one stipulation on this. I want you to try and solve this without using algebra, OK?

126.P: Yup. So again, I would separate the students into those pass and those fail. OK? And so then, among those who pass we have the average of 80. Among those who fail, the average grade those who fail—Oh. Okay, so I tend to think that the cut off is like 70? To pass or fail, I don't know what is the cut off, but OK. Anyway, I don't care, pass or fail, Group A: A+B are the total number of students. And for A group, the average is 80 and for the B group it is 64, and the two averages of both A and B is 68. Okay, so what percentage of class passed the uh—and I'm not allowed to use algebra to do that?

127.I: Well, I'd like you to try it without algebra, but if you need to write something down [puts pen in front of P] feel free to.

128.P: Let's say this would be A and B [draws horizontal line shown in figure 3] and this is A and B [labels below the line] and the average of this total [marks parentheses around endpoints of horizontal line] is somewhere here, say 68 [writes this above horizontal line]. And the average for only this group [runs pen under segment labeled B] would be like 64 [writes this above segment labeled B]. Average for

Group A is 80 [writes this above segment labeled A]. This 80 is so much larger than the 64 [pointed to 68], I would say 12 larger, and the average of Group B is only 4. So if the class average is closer to—it was 68 which is closer to 64. And the average, we can see 12 and 4. If you have 16 [writes this on paper] I would say 75% are in B category and 25% in A category. Just, without going through the more detailed algebra I can show—Uh x would be the A and [writes x and —] and then you solve those equations, but I just used this logical argument to look for the uh—

129.I: OK. So you were—let me make sure I understand what you were doing. You said that 12 and 4 together gave you 16, so you're saying that

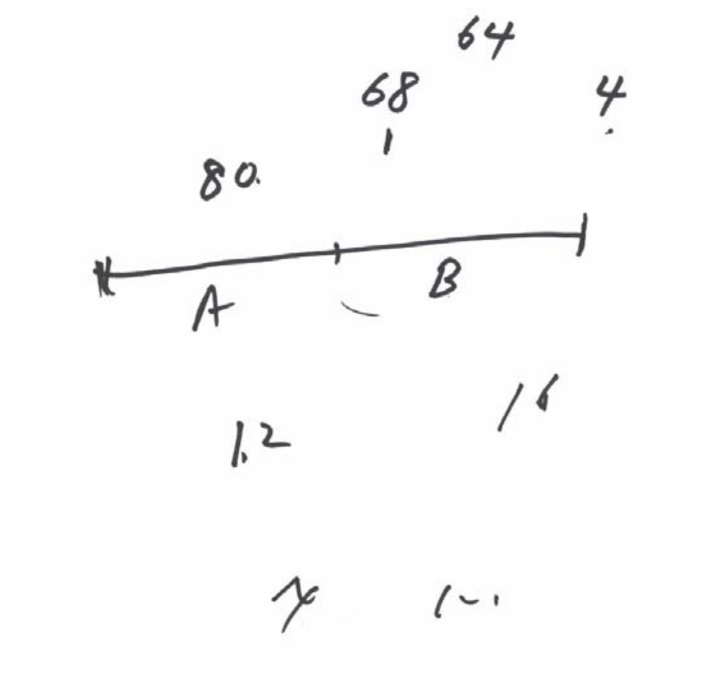


Figure 3

130.P: The deviations, the difference between the true average and the two individual average, that means that this difference is only four [spans finger from 68 to 64] but yet this difference is 12 [spans finger between 80 and 68] and in order for the true average to be 68 that means the B, those who only have 64 is a much higher number.

131.I: So you've got more—

132.P: More weight over there. And with this difference and the quick, the estimate would be—there one to three, the ratio, for the total; so 25 versus 75 percent.

133.I: Okay. I think I understood what you were doing there. Good. [Time 21:18] I have another, another—this one is one more—

134.P: More to try?

135.I: Oh yeah, more to do. I don't know if it's any more challenging, but more to do. All right, now this is not really a bar graph [puts paper with figure 4 on it on table in front of P]. Instead, what we're looking at is a city, and these are buildings [points to the bars on the bar graph].

136.P: Oh wow. OK.

137.I: So, this is building one and building two [points to bars in succession], building three—What I'd like you to do is show me what you think the average height of the buildings are. So, where would you say is the average building? And, draw a line where you think it might be and then tell me how you figured that out.

138.P: Mm-hmm, OK. Let me try to—the average, you have about seven buildings [points to bars in pairs of two] and two of them [points to 2 and 5] are considerable taller than the others. And I would, just to quickly average, like, this one, this one and this one [points to 6, 3, and 1], these three together have almost the average height of this one [points to 6]

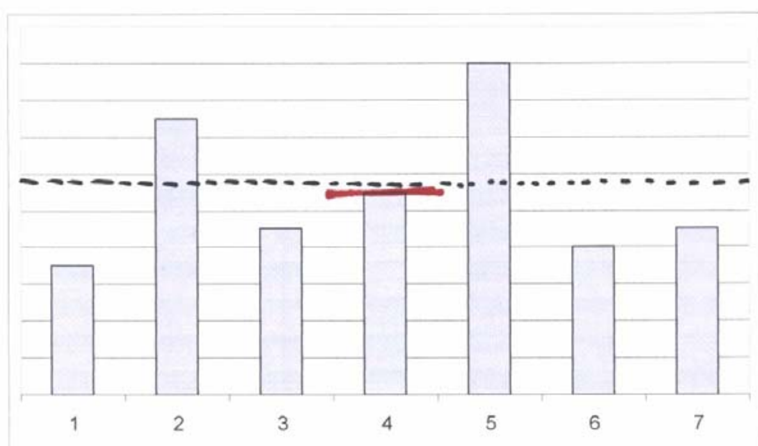


Figure 4

139.I: OK.

140.P: And in fact, I try to add up, get a quick [runs fingers along each bar] —Let's see, the three shortest buildings would have like an average height like this one [points to 6] And, the three tallest buildings [points to 2 and 5], well actually these two are closer [points to 4 and 7]—uh, let's say that I use this line as the possible average [points to the line on the graph that is one and a half lines above the top of bar 4] so this is one and a half above, from the, uh below [points to space between top of bar 4 and the proposed mean] this is a half and this is two above [points to top of bar 2 above proposed mean and then top of bar 5], so these three building would have an average of about this line [again pointing to the line proposed to be mean], about here. And but then I cannot ignore this one [points to bar 7] so if I include this one, that means that the average of the four [points to 1, 3, 6 and 7] would be somewhere here [points to just slightly above the top of bar 6], but there are three and four, the four would outweigh the three tallest buildings and if I take the average of these two [he has one finger on proposed mean for tall buildings and one finger on top of bar 6, he moves his fingers together as he says "take the average of these two"] it should be closer to this one here [pointing to bar 6] so I would say that the average would be below this line [points to line that is marked as a dotted line on figure 4].

141.I: [picks up black pen and draws dashed line on proposed mean] so you're saying—I'm just going to put a dashed line here, about right here?

142.P: Yeah, about—so let me just check it.

143.I: Once I do it you can tell me if you agree with that.

144.P: So, let me take another look. And if this one is almost [points to bar 4] is very close to this building. So, I would have one, two, three, four below this one [points to 1, 3, 6, and 7] and two that are well above this one [pointing to bars 2 and 5]. So if I use this [pointing to dotted line] as a reference for the height. And the two tall ones, would be one, two, three, four, and, uh less than five [counts the spaces between lines for top of bar five and then top of bar two – that is the spaces above the proposed mean] of the, uh, the unit. Above the average.

145.I: OK so that's above the average?

146.P: Yeah. So, those above this average is about a total of five. And then I can quickly add those below. And this would be like one and something [above bar 6] and this would be one plus something [above bar 7] and so it would be [waving fingers between bar 6 and 7 – the space from the top of the bar to the dashed line]—well, let me just try and see this is one or two [measuring space above bar 1] two together, this would be three, four [put finger over bar three, but then moved it to over bar 6] no, yeah, then three and four [above bar 6], then five and six [one finger over bar 3 and one finger over bar 7]. So, anyway, by adding those shorter than the average, this is not the—the off-estimate [pointing to dashed line]. I counted five and this [moves finger below the dotted line] this is six or seven. So, it should be a little bit lower than that [points to just below the black dotted line].

147.I: Okay, so what you were doing was—you were—correct me if I'm wrong. But, I think I was understanding that you were taking how far it was from the top of each bar to your estimated average. And you're trying to make what's between here and here [points to top of bar 6 and proposed mean] the same as between here and here [points to proposed mean and top of bar 5].

148.P: Exactly. —I'm allowed to? [has pen over the paper]

149.I: Oh yeah.

150.P: I could say, well this one [draws red line on figure 4] could be the average. If this is the average, then the, it would be, the—there would be two buildings, the tall ones, this one and this one [points to 2 and 5] and four below this one. But the two tall, then this building [points to top of 4 where new proposed mean is] would be taller as compared to those that are below. So if I add the one, two, three, uh, three probably point four [adding spaces above new proposed mean on bar 5] and this is probably one and two [pointing to spaces on bar 2 above mean] so 6.4. So, the two, the tall buildings we have combined, 6.4 unit tall. Of this average height. And the four shorter building, where this um might be like 1.6 [points to space above bar 6 to new mean], this would be very close to 1 [points to bar 7] so 2.6 and this is 3.6 or 3.7 [points above bar 3] and 5 and 6.7 [pointing above bar one]. So, the two are very close

151.I: Yes, very close

152.P: This building [points to bar 4] could be, the building in the middle could be the, well representative of the average height.

153.I: That's our average [points to building 4]. Okay.

154.P: It could, it looks that way [laughs].

155.I: So, when you were defining—when you were defining average before, you were saying that what you would do is you would add all of these up [points to each bar on picture] and you would —

156.P: Divide it by the number

157.I: —divide it by the number [moves arm horizontally across the bars to about where proposed new mean was placed] and that should give you about that height when you do that.

158.P: Exactly

159.I: But what—you didn't use that to solve that. Instead, what you were doing was you were taking above and below and you were making sure that they were the same.

160.P: Yeah, well, that—the way I did it is exactly the same way as if you add all of the height of the buildings divided by the number. But to do so is equivalent to divide each building by the total number of building and add up the difference and it should add up to zero. Okay, but let me clarify this. The difference between this mean and the median. Median, I would say, if I ask how many buildings are below this height or they're above this height what is the, what is the middle, the median height. And you see this one, two, three buildings are [points to 2, 4 and 5] are above, all above this height [puts finger in a place lower than the heights of those three buildings] And one, two, three, four are below this height [points to 7, 6, 3, and 1] and points to the proposed mean. But then you have four compared to three, so this [points to bar 3] would be the median because actually this is a little bit taller than this one [points to 7]. Well, depending on which ever one is taller. Then this [points to bar 3] would be the median height building among all the height of buildings under consideration. Anyway, these two are different [points to bar 4 and bar 3].

161.I: Yes, very much so.

162.P: Very much so, yeah, very different.

163.I: OK. Now, I've got another one [puts dot plot, figure 5 on table in front of P] I have lots in here, actually. This is—this is a dot plot and it really isn't important what it's of, but what is important to know is the mean of this set is right there [uses blue marker to place hash mark on figure at the mean.] And, do you see these red, darker dots? That means that there's more than you can see there. There's more dots there—you couldn't actually count up how many dots.

164.P: There's more there than is there.

165.I: There's more data points there than are actually shown [Time 28:33] by dots because it didn't fit on and I couldn't graph it any other way. My question is this: If I add one dot [picks up green pen and adds dot on column of dots that is two columns over from the mean] and I add it to right there, how is the mean going to change?

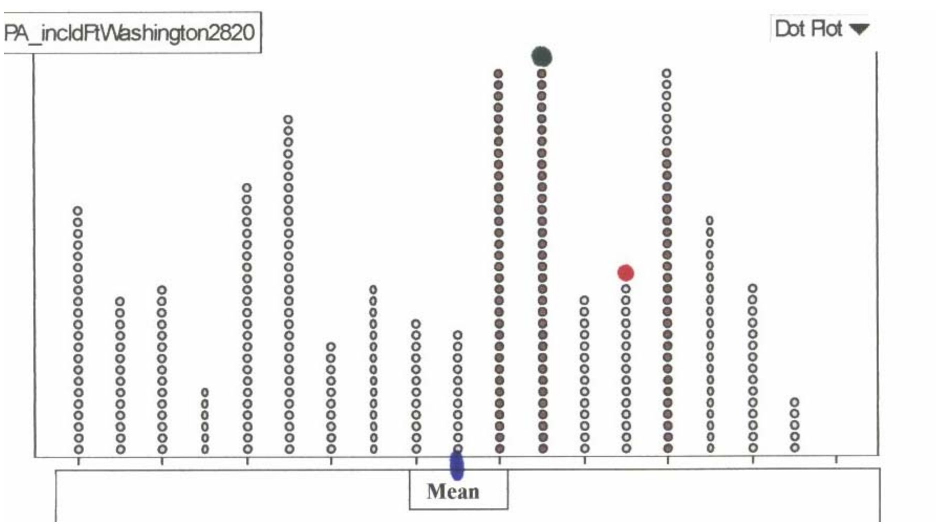


Figure 5

166.P: If you assume that the mean is the average of all the points—

167.I: mm-hmm, it is.

168.P: All the dots involved [spans finger across the dot plot] Okay?

169.I: Yes.

170.P: And the—because these three have a lot more that are involved [points to columns with darker dots in them] so just without counting it, just adding—I would—even without counting it I would presume that there would be more than hundred—?

171.I: Yeah.

172.P: Hundred data points. When you add one data point and also that data point is so close to the mean, it would not change the mean at all.

173.I: Yeah, oh, so you're saying—

174.P: It would change it but it would not be of any significant amount.

175.I: Okay, if it changed, even slightly significantly, which direction would it go?

176.P: It would increase it.

177.I: It would increase it?

178.P: Because, um, yeah. Well, first of all, the, assuming they are more than a hundred data points.

179.I: Yeah.

180.P: When you add one, that the weight would be only one percent.

181.I: Okay,

182.P: Okay, so adding one, there would be less than one percent. And the fact that this data point is close to the original mean that it would only slightly increase the original mean, but by much less than one percent.

183.I: Okay. Depending on how many there were—

184.P: But, no—yeah depending on how many there were, but also at this set of data points [points to column with added point] is much closer to the mean than most of the other data points.

185.I: Okay, I see what you're saying

186.P: Okay, so the effect would be probably much less than one percent.

187.I: Okay. What if I added—I'm picking another color—

188.P: At the far, far end?

189.I: Instead of adding the dot right there [points to green dot that is above 2nd column to the right], two away from the mean, what if I added it right here [places a red dot above the column that is 4 rows to the right of the mean]. How would that affect the mean?

190.P: It would, the same conclusion would apply but—again you're talking about one data point out of a large number, original number. But, since this new data point is further away from the mean, okay, it would have, it would again slightly increase the mean, but compared to the previous one we would have a larger, a slightly larger effect. That is further away from the mean [points to the right side of the dot plot].

C.1.4 Partial Transcript of Task Based Interview with B

5. I: OK. This first question; the data for this question came from the DASL website [places histogram in front of P, figure 1] and I made a histogram. And what this is is a histogram of the number of grams of protein in 77 boxes of cereal [P moves handout in front of herself]

6. P: OK, so [writes as he talks, figure 1] 77 boxes of cereal.

7. I: Correct

8. P: And -

9. I: Let me explain this to you. On the vertical axis, what we have is the count -

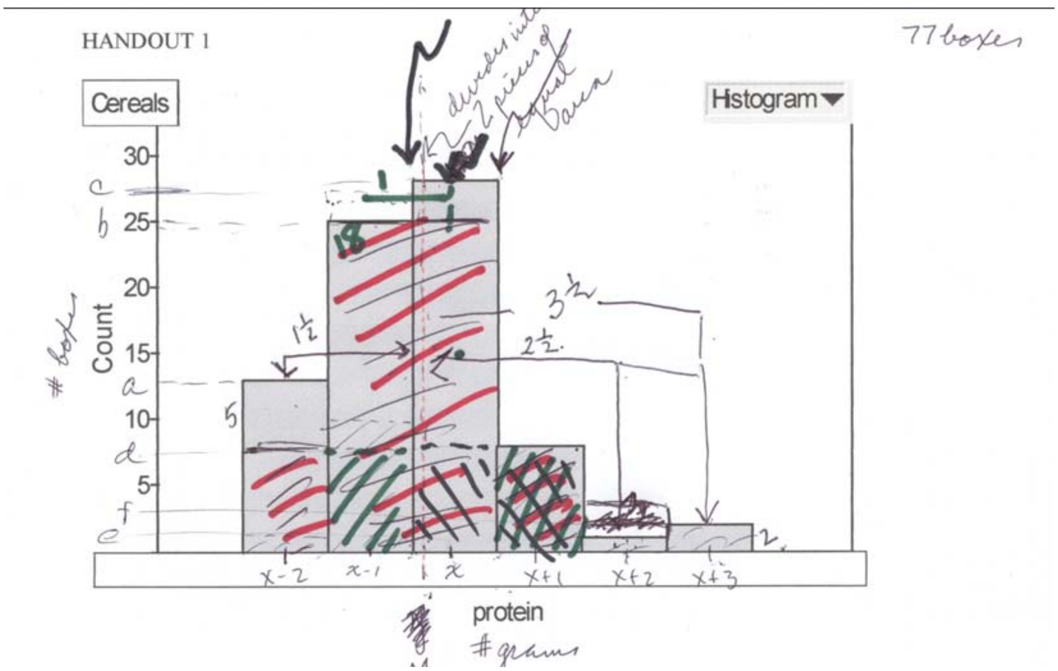


Figure 13

10. P: -the number of boxes?

11. I: Yes, this is the number of boxes of cereal [P labels this axis "# boxes"] and across the horizontal is the number of grams of protein in that box of cereal [P labels horizontal axis "# grams"]. So in this first bar there would be possibly twelve or thirteen boxes that had this count [points to first bar] of grams of protein. Now, I've hidden how many grams there are, but I can tell you that these are integer values. OK?

12. P: What are integer values? The -

13. I: OK, in other words, this histogram represents a particular integer value, such as four or eight, grams of protein.

14. P: So, a single integer value. So, it's not an interval?

15. I: A single integer. That’s correct, it’s not an interval

16. P: That’s odd

17. I: Yes, it is. It was an odd way of it being displayed. So, do you understand what’s displayed here? Do you have any questions about -

18. P: -[speaking over] I understand what I understand

19. I: OK. If you have questions that come up as I am asking other questions, feel free to ask, because that’s not the important part.

20. P: OK and this is [pointing to each bar of histogram] is equal spacing?

21. I: Equal spacing, yes.

22. P: Uniform spacing.

23. I: Uniform spacing, correct. The red dashed line [points to line on histogram] that’s on here, divides this into two pieces of equal area.

24. P: Mm – hmm, so—[draws a squiggly line from top of red dashed line and labels it “divides into 2 pieces of equal area”]

25. I: —[speaking over] Now do you agree with that placement, first of all, do you think that’s a fairly good estimate of the splitting the, into equal area? The histograms into equal area?

26. P: So the shaded part on the left side of the red line is the same as the shaded part on the right side?

27. I: That’s correct.

28. P: That’s your claim?

29. I: That’s my claim.

30. P: Well, let’s see. [Time 5:02] All right. So, I’ll [draws dashed line across second histogram bar, figure 2, then adds dashed line to first bar] That’s about that much [adds second dashed line to second histogram bar and then draws diagonal lines to fill in]. So, OK, those areas account for those [fills diagonal lines first under two dashed lines on first two bars, then on three right-most histogram bars]. So the claim is this area here is the sum of these two [points to third histogram bar that is not shaded and two parts of bar one and two that are not shaded]. I’d say “no”.

31. I: OK

32. P: [pause for about 5 seconds] Well, the sum of these two [points to unshaded parts of bar one and two] plus this [points to part of bar three that is to the left of the dashed line]. So, that is, if I put this on top of this [points to unshaded part of bar one and indicates moving it to bar two]. Well, let’s see, it would probably heighten this a bit [points between shaded part of bar two and unshaded part of bar one, then adds a dashed line just above bar two – see figure 2] so—[turns histogram and lines pen top next to histogram bar 1 non-shaded part]. Hmm, that makes it maybe a bit more [places another dashed line above bar two but a bit

higher]. So then the question is, whether those two [points to two bars' unshaded parts] plus this [points to part of bar three that is to the left of the red line], would that equal what's here [points to part of bar three to the right of red line]. I think it will be more than what is here [pointing to part of bar three to the right of red line].

33. I: OK. All right, so if it's more, you might move that line slightly to the left [indicates with hand]? Or to the right? To get it about equal area -.basically -

34. P: Well, if this is, if this is more on this side [moves hand to the left of the red line], then I think that that -

35. I: OK. Is there a statistical measure that is related to that line [points to red line on histogram]? That red dashed line.

36. P: Well, um, you mean one placed so that it is equal area?

37. I: Yes

38. P: Well, since these are integers [points to the hash mark at bottom of one histogram bar] I have to think about this a little bit differently. I don't know—I don't exactly know what you might be intimating by this. So, say this is some integer here [runs pen between left and right edge of bar three]. The fact that you've placed this over here, you're saying there's some statistical measure that's less than this integer. Is my assumption. Or, is the integer [again pointing pen between two edges] all the way across here? I don't know what you mean. Ok, since you haven't labeled this.

39. I: Right

40. P: And since you're creating this new sort of representation. What do you mean?

41. I: So, you would say at this point, because you don't information to say that that's a statistical measure or not. Is that, did I understand you correctly?

42. P: Well, OK. No, what I'm saying is because you've changed kind of what this represents [runs finger over third bar of the histogram] I don't know what you mean by this line [runs finger along the red dashed line] here.

43. I: OK

44. P: So, if this represents an integer, is it the whole space that represents the integer [runs finger between edges of the bar]? Or is it just the midpoint of that that represents the integer.

45. I: [picks up pen and marks the midpoint of the third bar] let's say that this [places and x under the midpoint hash mark] is the integer at x.

46. P: Oh, OK

47. I: OK?

48. P: At the midpoint of each [points to bar three]

49. I: At the midpoint here [places hash mark at midpoint of bar two and writes "x-1" under it] is another integer, we'll call it x – 1, just so you can get a feel for it.

50. P: OK, that’s more information.

51. I: OK.

52. P: All right, so, yeah, it would seem like that would be the median.

53. I: OK. Then why would you say that would be the median, then?

54. P: Well, I have [points to midpoint of bar 2, then bar 1, then moves pen from bottom to top of bar 2]. Let’s [10 second pause]—the median has to be that point [6 second pause] it has to be that point at which, from which half is to the left and half is to the right. Now, hmm, it doesn’t work, does it? Because, um, [5 second pause] if I only have these as possible values [points to midpoint hash marks drawn on bars 2 and 3], x minus 1 and x as possible values, I don’t have any values in-between as possible values, right? That’s what you’re saying?

55. I: Yes, that’s correct, that’s what I’m saying

56. P: For all these integers, OK. Well, then the median would at most be half way between x minus 1 and x [points to midpoints]. So the median could be, presumably along [runs finger along side between second and third bars] this line, but it couldn’t be someplace in between.

57. I: OK. So, it’s not the median?

58. P: [7 second pause] I have to think a second about this.

59. I: That’s fine.

60. P: Um, yeah.

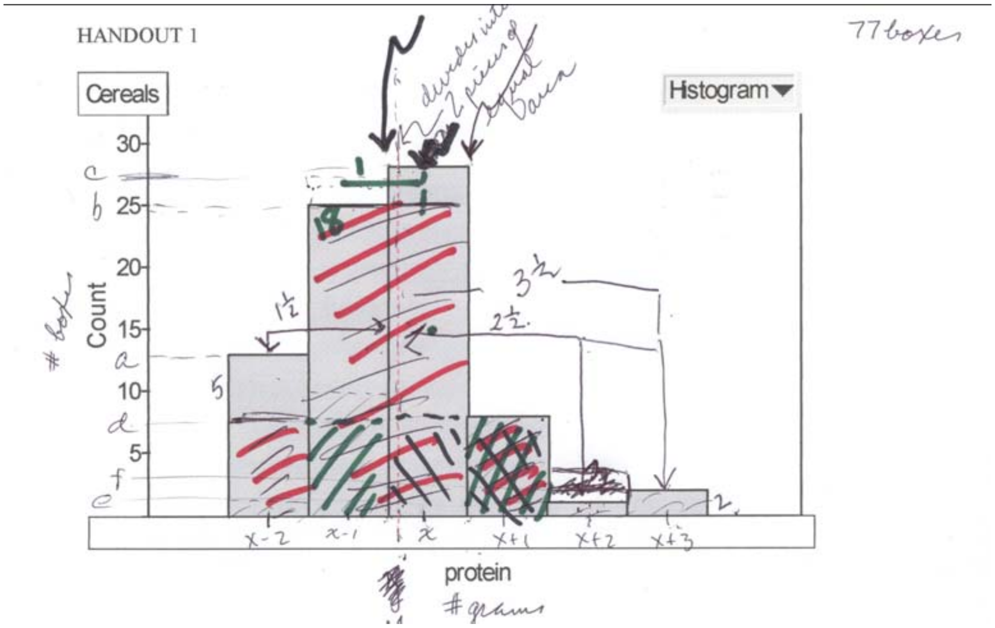


Figure 1

61. I: OK. Could you show me where you think the arithmetic mean might be on that histogram?

62. P: Well, Ok. [Time 10:10] Ok, so these are all the integers [points to hash marks at bottom of each bar of histogram] And we've got an x-1 and a x + 1 I assume [writes x – 2 below bar 1, x + 1 below bar four, x + 2 below bar five and x + 3 below bar six, figure 2]

63. I: They are uniform, so -

64. P: All right, so, um the distance—So I have some value [writes x, writes over it y, then crosses out and writes µ] well why don't we call it "m" for -for mean. Uh, So, I'm looking for the value of m such that -[begins to draw horizontal lines from the horizontal axis over to tops of bars] some "a", some "b" [labels the horizontal axis with these variables as he speaks] c [continues to draw horizontal lines to top of each bar and label with a variable] Ok, so we're looking for a value such that, uh, x plus three [writes as he speaks, figure 2] times e, plus x plus, or times f, sorry -wait, is that right?

65. I: Yeah, I think that was f. That's right.

66. P: Well, you got the idea. X plus 2 times e plus x plus one times d [8 second pause while he isn't writing] -I'm trying to decide if I want to do it this way.

67. I: OK

68. P: OK, so, um, I'll show you in a second. So x plus one times d, plus x times c, plus x minus 1 times b plus x minus 2 times a -I think that's generally -.divided by [writes line and then a + b + c + d + e + f"] Ok, so that's what m is.

69. I: OK, uh

70. P: Um [pause for 3 seconds]. Or—but that doesn't tell me anything. I mean, it's just a formula. I'm looking for a value m such that the distances on either side of m [points pen to left and right of where he has labeled m on the histogram] ok? Uh, sum the same. And, I don't know where m is. So, the easier way to think about that—I mean, I don't know which of these points [points to bars on histogram] would be to the left and which would be to the right.

$$M = \frac{(x+3)f + (x+2)e + (x+1)d + xc + (x-1)b + (x-2)a}{a+b+c+d+e+f}$$

$$a(M - \cancel{a}(x-2)) + b(M-(x-1)) + \cdots + f(M-(x+3 = 0$$

Figure 14

71. I: OK

72. P: To the right of m, theoretically. And so, I'm looking for a point such that when I do my subtraction of those points from m, I get zero. So, um [begins to write second equation shown in figure 2 as he speaks] x minus two times a. So m [crosses out a that has been written] m minus x minus 2 and there are a of them [puts parenthesis around m – (x – 2) and writes a] plus b times m minus x minus 1 plus and so on through f times m minus x plus 3 has to equal zero

73. I: OK.

74. P: So -[pauses for about 5 seconds]

75. I: Help me understand what you just did with this formula [points to second equation]. What you're saying then is that for each count in one of these bars [points to first bar of histogram], that's the letter; that's the count.

76. P: Mm-hmm, yeah.

77. I: That you're going to take whatever your mean is minus what that is [points to midpoint of bar 1] and when you sum all of those up, that should equal zero.

78. P: Right.

79. I: OK. I'm just making sure that I understood your formula. Is there any way, with the information I've given you that you can make an estimate without [Time 14:59] using a formula?

80. P: Well—[about 16 second pause] I can -well presuming it is someplace in here [draws a dashed line inside the third bar of the histogram] the problem is if it's not—Ok, well um, I'm going to at this point work as if it's right here [marks curved arrow on top of histogram, figure 3, to point to dividing line between bar two and bar three].

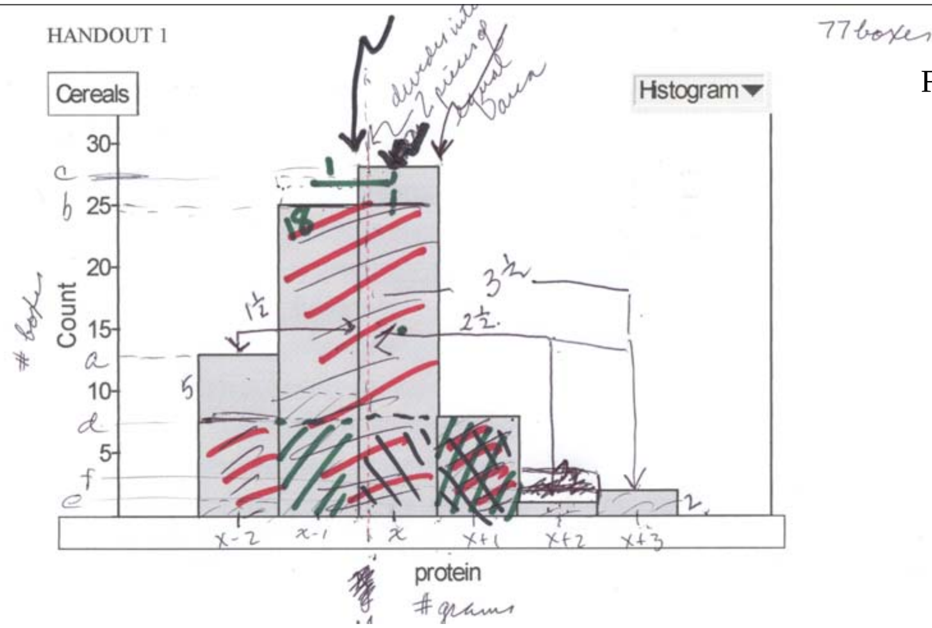


Figure 15

81. I: On the dividing line between the -

82. P: And make some adjustments after that, yeah.

83. I: OK

84. P: So, if that's true, then this part would balance out this part [marks diagonal lines through histogram bar two and draws a horizontal line even with top of bar two on bar three and draws diagonal lines through this lower portion of bar three – in red in figure 3]

85. I: OK

86. P: And this part would balance out this part [points to bar four and then draws a horizontal line across bar one and shades both parts in – again in red]. OK so I'm talking about these three parts these three parts [points to unshaded part on top of first bar, and bars labeled "x + 2" and "x + 3] someway balancing—now that's—its distance from here, it's about one -however you want to count it, away. And, this is about two away [points to bar labeled "x + 2"] so, it would count for twice what it is [draws line that is even with last bar on the right, later in interview crosses it out]. This is about three away [points to last bar on right] so it would

count for -did I do that right? Here [points to dividing line between bar two and three – the proposed mean], all right [points to midpoint of bar labeled “x+1”] so to move it, to move it -ahh, no [crosses out line above bar labeled “x+2”] let’s do this again. I’m going to use your colors

87. I: Yeah -there’s a good reason for colors

88. P: [begins to shade in bars two and three as described above with color this time, then shades bars one and four] All right. So -[pauses for about 13 seconds]

89. I: Can you explain in words what you’re trying to do right now?

90. P: I’m trying to figure out what something [taps pen over bar one, the un-shaded part] on this side amounts to [moves pen to over the far right bars] on, um, what it would balance on this side.

91. I: OK -and -

92. P: So, it’s distance -the distance from here to here [points from midpoint of bar labeled “x+2” to the proposed mean] um, times the weight, um, would need to balance this distance [points from bar one to the proposed mean]. And the question is distance from where to where.

93. I: OK. All right, I understand

94. P: So, I was assuming that this was, that that’s [draws a bar from midpoint of bar one to the proposed mean] the distance. And all of my values are right here [points to midpoint of bar one] and my distance, real distance is right here [points to line just drawn].

95. I: OK

96. P: So, this real distance [places pen over bar labeled “x +2”, then draws a horizontal line from the proposed mean over to the midpoint of this bar] is right there and now, so um—I could say that this is something like 2 and a half; a distance of 2 and a half [writes 2 ½ over horizontal line just drawn] and a weight right now of, I don’t know, one [writes a “1” over bar labeled “x+2”]. And this is a distance of [moves pen to over the horizontal line drawn from the first bar to proposed mean] one and a half [writes 1 ½ above line] and a weight of -let’s see -maybe five? [writes “5” next to bar] Ok, so this product is 7 ½ [pointing to line on left side of histogram] and 2 ½ times—that would need to be 3 [crosses out 1 just written and writes 3 over it] to balance that out.

97. I: Why did -OK, I see what you were saying—it would have to have to be three to balance that out [emphasizes word “have”]

98. P: Yeah

99. I: OK

100. P: So, um, in other words, this counts [runs pen over bar labeled “x+2”] for [begins to draw a line above this bar even with about 3], it’s only one -

101.I: Right

102.P: OK [talking to self] times two and half, so it would have to be three -So, I've, anyhow, I've got -here I've got [runs pen from proposed mean to the left] seven and a half—I need to account for [moves pen to right] 2 ½ I've accounted for here -Here [now draws an addition to horizontal line on right to over the midpoint of the rightmost bar] this would be a distance of [writes 3 ½ about this line] three and a half and a weight of -it looks like about two [writes 2 beside the bar]. All right, so that's seven [pointing to rightmost bar and horizontal line] and this is, what did I say, that would be -

103.I: I think you said two and a half

104.P: two and a half -.so that's nine and a half. So, uh, what was the question I was trying to answer? [laughs] Um, what's the mean?

105.I: Right.

106.P: OK, so as I'm calculating it, there's actually more weight on this side [putting hand on the right side of the histogram] then on that side, so -well, that makes sense because it wasn't right at the bar.

107.I: Ok

108.P: Um, now, what if I [draws a squiggly line from dividing line between bar three and four] said the mean -well, I'm pretty sure—pretty sure it's more than half way -so what if I said the mean was here [places a squiggly line from midpoint of third bar up]?

109.I: Can I ask you why you think, you're pretty sure it's more than halfway?

110.P: Cause um, well—[8 second pause] well, uh, maybe it's not [crosses out squiggly line he just drew] Let me do some more thinking there. That's the old, "ask a question and get someone to change their mind" there.

111.I: Wasn't my intent [laughs]

112.P: I know, I know—[straightens paper and looks at again] OK, well let me, let me verify for myself that it's not there [pointing to dividing line between third and fourth bar]. So, supposing it were there. Then, as I'm balancing things I -do you have another one of these? [asking for another histogram]

113.I: You know what, if you want to we can just kind of [places a blank paper on top of histogram] put it over. Can you see through there well enough?

114.P: No.

115.I: No, too bad -OK, well go with a different color then.

116.P: OK, All right, so if it were here [pointing to dividing line between bar three and bar four] then I'd be trying to, oh shoot, it's obvious [says this after quickly running hand across the bottom of the histogram] Um -

117.I: Why is it obvious?

118.P: Well, OK, if it were here [points to new proposed mean] this part here [begins to shade in fourth bar] would balance off [draws dotted line even with top of bar four across bottom of bar three and shades below it] that part. OK. Now, this part here [points to fifth bar] would be, ok, a distance of one and a half, one and a half? Yeah, one and a half and a weight of one versus, well these would be a distance of zero, so [points to bar three] these will be a distance [now pointing to second bar] of, uh, one and a half times, you know, a lot more.

119.I: OK—all right -

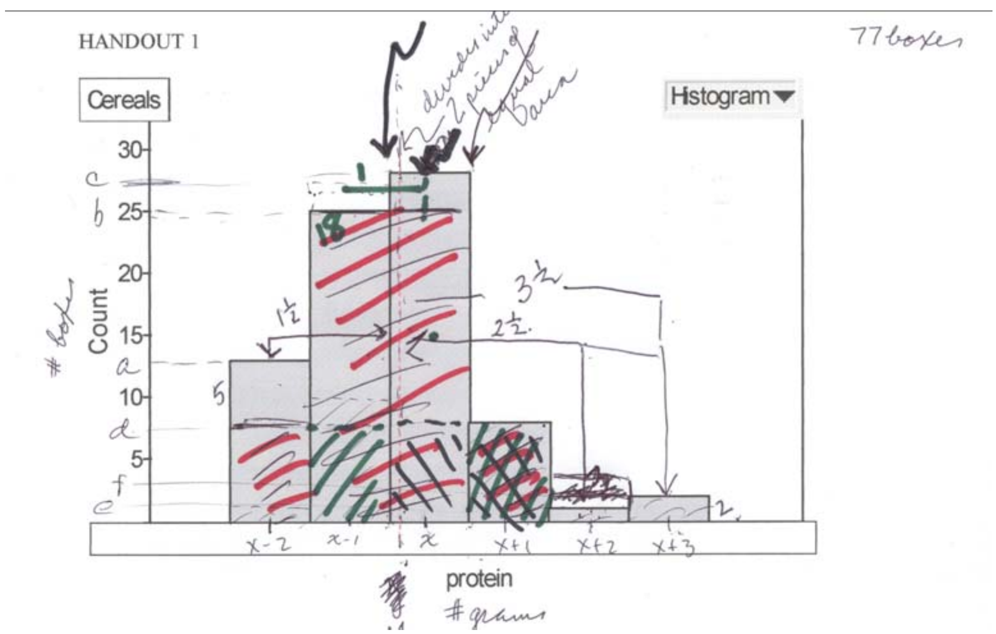


Figure 3

120.P: So, it's not there—[points to line dividing bar three and bar four]. So -and it's to the right of this [points to line dividing bar two and bar three]. OK, now, how about [redraws line above midpoint of bar three] midway. How about right at that point [points to x labeling the midpoint], right there? Well, um, course then, these two [points to right half and left half of bar three] balance each other out. And then, uh, probably going to want that [talking about color pen] OK, so right now I'm looking at this [draws a dotted vertical line down middle of bar three, Figure 3 in green]. Um, ok, so those two balance each other out and this [draws green lines across bar four] balances out that [draws an equivalent portion on bottom of bar two and shades in green] and this [spans fingers from top of shaded in part of bar two to top of bar] is one unit away, right? And that's one [draws horizontal green line and labels "1"] and a weight of, oh, I'm thinking maybe 18? So that's about 18 and -I'm here right [places pen over green vertical line]? OK, and this is [pointing to the fifth bar – 8 second pause] Ok, so I'm going from here [points to midpoint of bar three to midpoint of bar five] to here, right? So, this one is two units away and a weight of one, two times one -and this is [points to sixth bar] is three units away, one, two, three [points from midpoint of bar three and counts midpoints over to bar six] and a weight of two, so that's uh, six. And six and one is seven versus an eighteen.

121.I: So, you would continue to work this until you could find an approximation

122.P: Yeah.

123.I: OK, and let me just see if I'm summarizing—basically, what you were doing was, you're hunting for that point where you can, taking both distance and count into account, can balance those two out.

124.P: Mm-hmm

125.I: OK. All right, thank you. Um, for this next one I'm might ask you to -well, I am going to ask you to write something on a blank sheet of paper, but will start without writing. We'll just start with talking [TIME 24:56] and that is -suppose there was—a median had been discovered and talked about and used a lot and mode was, but arithmetic mean had never been invented. Nobody had come up with what arithmetic mean was. Do you think you would invent it?

126.P: That I would invent it?

127.I: Well, maybe not you -but do you think that there is a good reason for it to be invented?

128.P: Um -what good is it beyond median and mean -or and mode? Well, lots of people claim that there's no use for mode. Uh, so there's -

129.I: So, we'll just say median.

130.P: So, median. Ah -[8 second pause] well, I mean, it takes into account outliers in ways that the median doesn't and so if you're kind of concerned about the whole group and the values for the whole group you may, you know, mean becomes more useful. Uh, there's certainly a lot more statistical tests, or a lot more popular statistical tests that test, you know, whether two means are equal. Um, very popular in terms of any kind of comparison, you know, except for maybe basketball teams.

131.I: [laughs] now -

132.P: The mean income for geography graduates from a certain year from NC State or wherever Michael Jordan was from -

133.I: Right—

134.P: Is very high -Although, the mode -is probably quite a bit lower [a small laugh with that statement]. Um, so yeah, I'm fairly convinced that people have seen it as measure that does -well -OK -in terms of statistics, my guess is, since it is arithmetically easy, that it may be easy to work in the formulas and that sort of thing.

135.I: OK

136.P: Uh -now, if we had test scores from kids. Suppose we had NAEP test scores from kids. And we were comparing the modes instead of the means, in a sense. So the mode score went -what if the mode score went up -was what the headline said. Um -that, that might actually have more meaning, but the public wouldn't understand it as well.

137.I: Why do you think the public understands mean better than mode; given that mode is a fairly simple thing to figure out?

138.P: I think it's just what we're used to hearing. I don't think conceptually it's -

139.I: Is there something conceptually that makes mean more useful for comparisons? Since we do seem to use it for comparisons.

140.P: Well that's what I'm trying to think about it. This is, what if we were operating in a world without a mean. I would like to operate in a world without "means" [laughs].Um

141.I: Too many plays on words -

142.P: Yeah [8 second pause] yeah, I mean the basic thing is, is it important for us to be able to provide measures that in some way account for outliers. And, uh—and I think if we don't account for outliers -well, OK, if we don't -a couple of things. If we don't account for outliers, we don't get a measure of the whole in the same way. So, suppose the government were worried about the average tax that somebody paid. What if they reported the mode, or what if they considered the mode and not the mean? Well, they wouldn't bring in the same amount of money, it's not predictable. I mean, the mean is going to be -uh the mean of k values is going to be the same as the product of -I mean, the mean of k values times the number, times k, is going to be the same as the sum of those k values. So, uh, so that then gives us a better measure of what the total is. You can't get that measure from mode. So, I suppose if you're concerned about equity, mode is better and maybe if you're concerned about kind of, accumulation and totals, maybe mean is better.

143.I: OK. [Time 30:30] All right, you've uh, taught on a number of—for a number of years on different levels. Um, is there something about arithmetic mean that you recognize in students as, they struggle with this particular concept? Is there something with that concept that you say, "students tend to struggle with that part of arithmetic mean"?

144.P: I haven't really taught a lot about it, so -

145.I: OK

146.P: OK, so, I'm kind of imagining what students do with this. Uh -

147.I: If you haven't taught it enough to answer the question, that's fine.

148.P: Yeah -

149.I: Yeah, OK

150.P: I don't think I have real experience to draw on—I can invent an answer -

151.I: Yeah, I don't think I need an invented answer. I can invent answers too [laughs]—um, I'm trying not to do that. Let me go on to the next question. Without using the algorithm of sum everything up and divide by how many you have, could you define arithmetic mean for me? And, would you write that definition down once you've kind of get to something that you say, that's a pretty good definition?

152.P: [Time 31:33] well I can define it in a number of different ways.

153.I: That's what I meant -

154.P: What do you want?

155.I: I would like to see as many ways as you would like to define it, since you say there are a number of ways.

156.P: OK, so I want the mean of [begins to write as he talks, figure 4] xi, such that it goes up to n, lets say it's finite. All right, so um, and we'll call that just M sub xi and so one way would be the sum of xi and i goes from one to n divided by n.

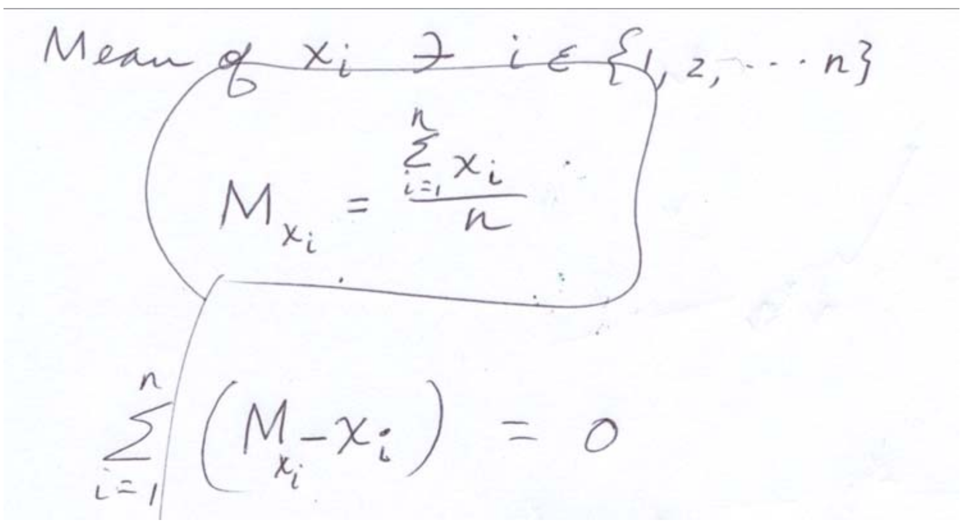


Figure 16

157.I: Which is the one way I don't want it defined

158.P: OK. Is there another way you wanted it? [laughs]

159.I: [laughing] could be -

160.P: Um -OK, so, let's see, so it's the -I don't know how else you might—I mean there are relationships involving it. I don't know what you mean by a definition, though.

161.I: OK. Go ahead and describe the relationships instead of coming up with a technical definition. Go ahead and describe those relationships that would represent mean.

162.P: So, I know that the mean [begins to write the equation at the bottom of figure 4] minus each value, this sum, as i goes from one to n, I know that that's zero.

163.I: OK.

164.P: Um, I know that if the mean [begins to write inequalities, Figure 5] is situated between x sub j and x sub j plus one that the sum as, let's see -. [Doesn't speak but writes the equation shown on second line of figure 5] woops, [crosses out j at top of second summation] goes from j to n. I know that's true [underlines equation in air]. Um, I know that—[draws horizontal line] you know, that if I have [places x under hash and dots to right, figure 6]

If $x_j < M_{x_i} < x_{j+1}$

$$\sum_{i=1}^{j} |M - x_i| = \sum_{i=j}^{\cancel{j}n} |M - x_i|$$

Figure 17

165.P: The xi's situated at different spots on a number line -there may be several of them at different places—I have [draws large upside down "v"] you know, a mean [writes M above] , although this is the same [waves hand to equation above, figure 5] you know, any number of those. I know that the distance [draws a horizontal line from mean to a point] here, times the number of weights here [points to x sub 3] plus the distance here [draws line from mean to place where x4 and x5 are] times the number of weights there is going to be equal [waves pen to other side of mean] on either side -

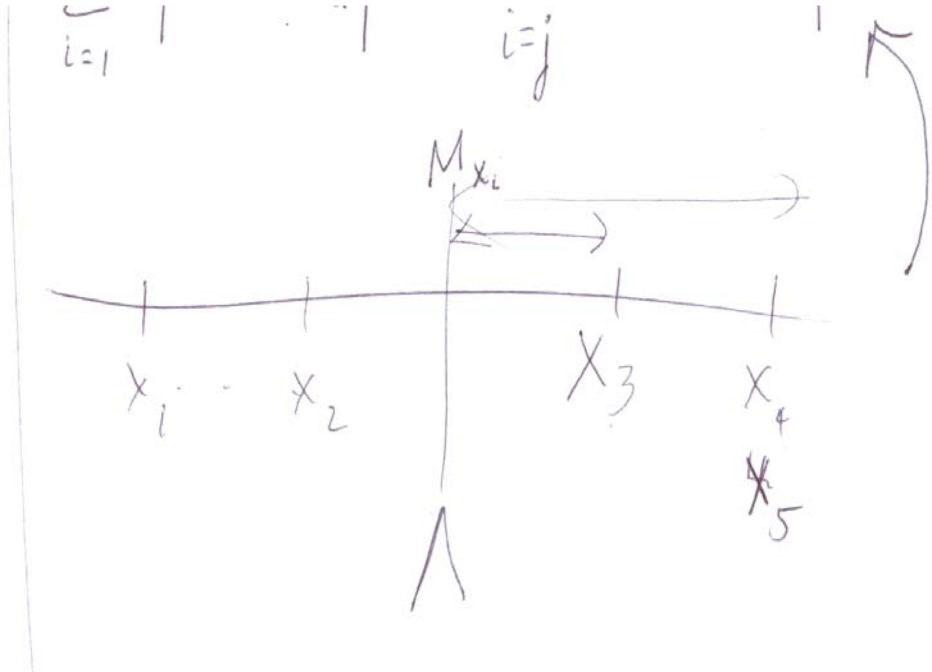


Figure 18

166.I: OK, mm-hmm [TIME 34:56]

167.P: which is equivalent to [draws arrow from number line up to formula in figure 5] that one.

168.I: OK.

169.P: Uh -[about a 10 second pause]

170.I: that -.that's a sufficient way to get started. I'm going to hold on to this because we're probably going to come back to it. So, I'll just set it there for right no [putting paper that was written on aside]. Um, I'd like you to try a task that, um, actually, I've adapted it from something Dr. Grayson Wheatley had put in one of his problems. But, if you would read that and if you have any questions about that -would you read that aloud and then if you have any questions I'll answer them as best I can.

171.P: [reading] "A class of students took a test. The class average on the test was 68" I assume mean

172.I: Yes, arithmetic mean.

173.P: Uh, "the average grade of the students who passed was 80 and the average grade of the students who failed was 64—[pauses two seconds]—What percentage of the class passed?"

174.I: Now, without using algebra, could you solve that problem and explain to me as you're solving how you're doing that?

175.P: OK. [5 second pause]. So we know—I don't know what you mean by "not using algebra"

176.I: I would like you to try and solve this without um, using variables as replacements, uh -

177.P: OK—all right, so if I took all the test scores and added them all up that's not algebra, right? [laughs]

178.I: That's correct.

179.P: All right. That means that I could replace every one of those student scores by a 68 [writes columnar list of 68's, figure 7]. Uh, those who passed was 80 and those who -OK —Or, I could take this set; and there's a bunch of eighty's and a bunch of 64's [writes down second column of numbers, figure 7] and that would be, that would be an equivalent replacement, in a sense—we would get the same sum.

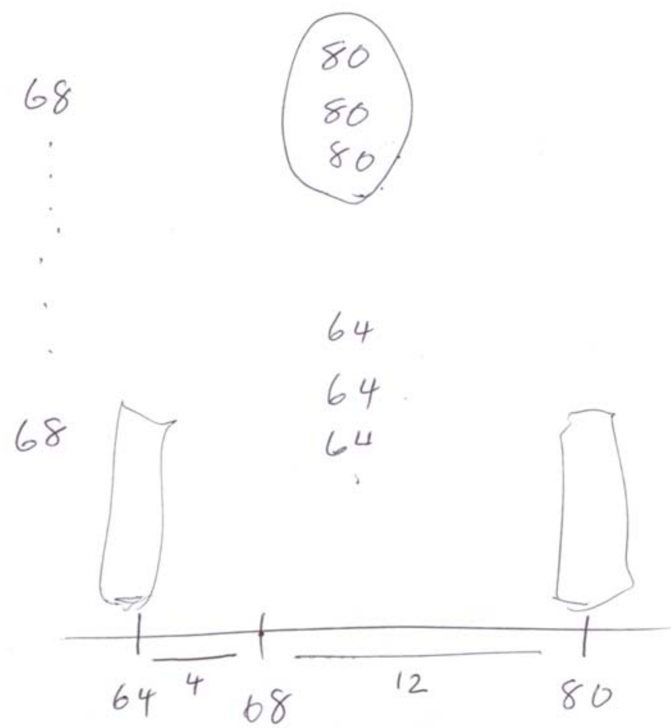


Figure 19

180.I: OK

181.P: Uh, as if I had kept the original scores, I had all 68's. So, the question is to get what percent passed [circles column of 80's], you're looking at how many of these out of the total are 80's. So [draws horizontal line across bottom of the page] you're looking at uh, 64 [adds hash mark and labels 64, then adds hash mark on right and labels 80] and an 80 and a 68 [draws hash mark and labels 68]. And you're saying [draws rectangle above 80] how many weights would you have to have here [draws rectangle above 64] to balance out the number of weights here to get it to be a mean of 68.

182.I: OK

183.P: So, distance here is 4 [draws line between 64 and 68 and labels 4] and the distance here is twenty, no twelve [draws a line between 68 and 80 and labels 12] ok, so that's three times. Yeah, OK. So, I would need three times as [pointing to weight above 80] sorry [moves to weight drawn above 64] three times as many here as here. And, if I had three times as many here as here [points from weight above 64 to weight above 80] then the total would be four units and three out of those four would be failing. So 25% passed.

184.I: OK. So you got 25%. Now, when you were using your definition up here—how do you see this definition [points to paper with definitions written out – figures 4-6] informing you for this answer. For solving this problem.

185.P: Well, it's this [points to figure 5, shown below]

If $x_j < M_{x_i} < x_{j+1}$

$$\sum_{i=1}^{j} |M - x_i| = \sum_{i=j}^{n} |M - x_i|$$

Figure 5

186.I: But, you didn't use algebra—this is very algebraic in nature.

187.P: From your perspective

188.I: Yes. And from your perspective, what was it about this algebraic form that you used that you used down here [pointing to sheet with class average, figure 7] when you solved this problem?

189.P: It's that the sum of the weighted distances is the same on either side -so that's what this is [pointing to figure 7]

190.I: So, this is basically [pointing to figure drawn in figure 7], what you've done is you've just taken this [points to equation, figure 5] and moved it here and found a solution.

191.P: Or, I have a concept that is represented here [points to figure 5] and here [points to figure 7].

192.I: OK. All right, thank you. The following picture here, this is -actually you can think of this A, B, C, D -these numbers are just numbers of heights, OK? It's not an actual number—pretend these are buildings. OK?

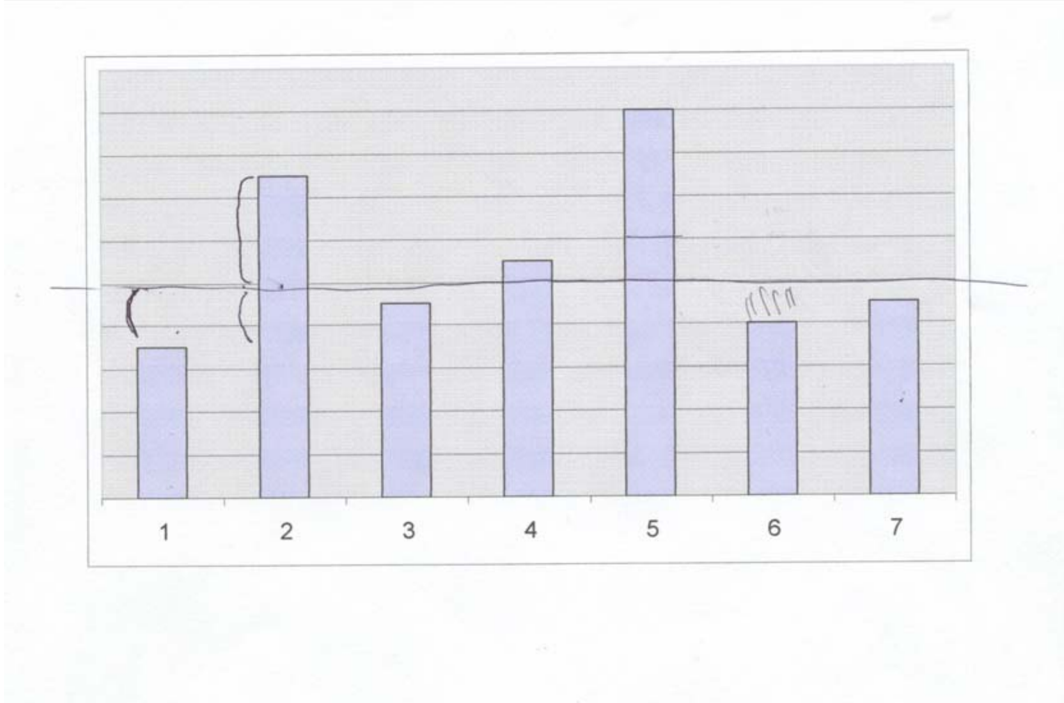
193.P: All right

194.I: These are buildings and so this is building 1 [points to bar 1] and this is building 2 [points to bar 2] and -

195.P: So these are labels.

196.I: Yes, these are just labels, thank you. That was the word I was looking for [laughs]. So, these are just building labels. And suppose that we were just looking at this. How could you tell, could you figure out what the average height of the buildings, the seven buildings were. And, could you draw a line where you think the average heights of those buildings are? [Time 40:22]

Figure 20



197.P: OK. Well, again, I could do some balancing and so what I would want is a line that would cut off [6 second pause as he looks at paper] um -well, [2 second pause] yeah. I want a line so that the part that's cut off above, I could transfer to whatever places below [is pointing to bars that a shorter]. In other words, I want to level it.

198.I: OK. Can you show me with one of them what you mean? You don't have to do the whole thing but could you show me -?

199.P: All right, so suppose my line was here [draws horizontal line shown below, figure 8]

200.I: OK

201.P: OK? And then what I'm saying is that this part [circles the part of bar 5 above line drawn] whatever that distance is [lays pen beside bar to measure] -Actually the line would probably be -higher. [laughs] Ok, whatever this part is [again lays pen next to top part of bar 5] I would transfer that, say, -yeah there's about that much, there [marks one unit on bar 5 above horizontal line and shades in one unit above bar 6]. There's three more there [runs pen along top three units of bar 5] Well, there's not one -one, two [has pen over bar one and bar 3, spanning from top of bar to horizontal line] -So, clearly it's higher.

202.I: OK

203.P: But, anyhow

204.I: But that’s what you mean -is you’re going to take a piece from up here [spans finger from horizontal line to top of bar 5] and move it into this space down here [spans finger from top of bar 6 to horizontal line drawn].

205.P: Yeah, yeah

206.I: And your end goal is what when you do that?

207.P: My angle?

208.I: Yeah, oh, the “end goal” that you have.

209.P: Oh [laughs]

210.I: [laughs, too] In English—sorry

211.P: OK, is to find a line so that when I do that movement [moves hand between lower part of line an upper part of line] all of these end up with a height exactly on that line [points to bars then along horizontal line drawn]. That would be the average height.

212.I: OK. Now, how does this doing this transposing of a piece here to the space here [spans finger from horizontal line to top of bar 2, then moves span from top of bar 1 to horizontal line] relate to this definition you have over here [points to paper with definitions on it]

213.P: I don’t think I’ve accounted for that in these [referring to written definitions]. Because what you’re doing is your, for these [points to left side of equation, figure 5] you’re taking some part of that distance and moving it some place else [now points to graph]

$$\text{If } x_j < M_{x_i} < x_{j+1}$$

$$\sum_{i=1}^{j} |M - x_i| = \sum_{i=j}^{n} |M - x_i|$$

Figure 5

214.I: Ok

215.P: So, um, I could probably make up something here

216.I: [laughs] why don’t you make up something there.

217.P: All right. So, let’s see [is running finger along definitions already written] I want [6 second pause] I think [7 second pause with finger under the equation in figure 4] its closest to this [circles as shown and draws a line down to the bottom of the page].

Mean of $x_i \ni i \in \{1, 2, \cdots n\}$

$$M_{x_i} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\sum_{i=1}^{n} \left(M_{x_i} - x_i\right) = 0$$

Figure 4

218.I: OK, so it's close to that

219.P: Because I think what I could say [writes equation shown in figure 9] all right, when I moved it, all right [now points back to bar graph and drawn horizontal line] then I've got my average height which is this [points from top of bar 1 to horizontal line] times n which is number of buildings [runs pen along each bar]. Is the same as the sum of [runs pen up each bar] each of these heights. It doesn't tell me exactly that I'm taking this piece and moving it here [spans finger along bar 5's marked unit and top of bar 6] but that what that would mean [begins to write second part of figure 9] is within here [starts with summation notation] what I'm doing is I'm taking [crosses this notation out] well let's just take x1 and taking something away from x1 and adding that to say, an x2.

$$n\,M_{x_i} = \sum_{i=1}^{n} x_i$$

$$\cancel{\sum_{i=1}^{n}} x_1 - a + x_2 + a + x_3 - b$$

$$+ x_3 + b \cdots$$

Figure 21

220.I: OK

221.P: And so on, so I keep the x1 plus x2 through n -

222.I: Right.

223.P: [continues to add to expression above] um, yeah, I mean, you know I could go on, but that's the idea.

224.I: Yeah, OK. And, in your thinking of this side has to balance out with this side [pointing to equation from figure 5] this has to equal this, is there any way that you can see that in this [pointing to the bar graph] picture.

225.P: Yeah, yeah. Because, what you'd be—All right, I have to think about this a little bit. [Is holding bar graph at 90 degree angle from how it was printed] so the m minus the x, I'm doing, you're saying the absolute values here [pointing to equation in figure 5]?

226.I: mm-hmm

227.P: OK. So we're just talking about the distance here [draws a line on bar two that spans from the horizontal line to the top of the bar] needing to equal the distance from here to, you know, in this case, here [draws from horizontal line down to top of bar 1].

228.I: OK. And you're saying if I sum -if this line were in the right place and I sum these [places hand over parts of bars above horizontal line] distances, these absolute distances, then that should balance with these distances [pointing to spaces from top of shorter bars up to horizontal line]

229.P: Which is the difference between the top and the line.

C.1.5 Partial Transcript of Task Based Interview with K

9. I: OK, here we go. [Time 1:11] On this sheet right here [places figure 1 on table in front of P] I have five grades of a student from different assignments. Could you explain to me in words how you would find the average grade for that student?

10. P: I would add up all the numbers [points to each number from top to bottom] and divide by the number, the total number of numbers.

11. I: Is there – if you were not allowed to do that. If I said you can't add them up and divide by 5, is there any other way that you can think of that can help you find the arithmetic mean?

12. P: [5 second pause] If I can't add them up?

13. I: Yeah

14. P: All right. I my head, I guess I would, by process of elimination, look for the middle value and kind of "weight" things.

First Grade:	90
Second Grade:	73
Third Grade:	80
Fourth Grade:	85
Fifth Grade:	97

Figure 1

15. I: OK. Can you explain to me, with those numbers [points to figure 1] what you're talking about?

16. P: All right. I see that this is the middle value [points to the 80], so I know that – I see that [laughs and now points to the 85], I see that this is the middle value and this student has one grade 5 points below and one grade 7, and 12 points below. So then, I would balance the distance below with the distance above.

17. I: OK. So you would be, saying I look and say maybe this number [points to the 85]

18. P: Right

19. I: And then you were finding how far away from below it and how far above. And your end goal was to do what?

20. P: Have the same weight on both sides of the number.

21. I: OK

22. P: So, I would adjust [points to the 85] this number up or down.

23. I: OK, so you had said that it was 5 and 12, so there is 17 below? And you would check to see if that was – similar to the other side?

24. P: Right. 5 and, oh yeah! [points from 85 to 90 and then 85 to 97]. And there it is.

25. I: There it is, yeah. How about that? OK, and let me ask you this. Can you describe how that is similar to adding everything up and dividing by 5? What you did, is there any way you can describe how what you just did, where you got the points below and above is the same as, or is similar to – since you're finding the same number both ways.

26. P: Um, let's see [3 second pause]. I would say, adding them up and dividing by 5 is just a short cut to doing it the weighting way.

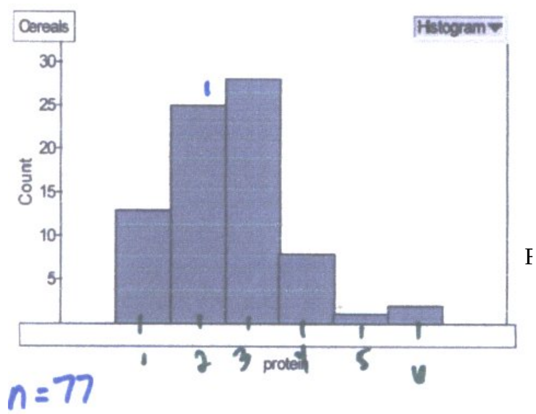
27. I: OK. All right. This number, let's say you said that's it [points to the 85], you mean the 85

28. P: Yes.

29. I: What does that number mean when you are looking at those five grades and you say the average is 85? What does that mean?

30. P: Um, it’s an, um, approximation of how well the student has typically performed on the assignment.

31. I: OK. Thanks. Got another one [Time 4:20] for you. This one’s a fun one because of the way that it is actually drawn [places figure 2 on table in front of P]. So, let me make sure you understand it before I ask you any questions. This is, I got data off of a free use website and what the data was is boxes of cereal. And across the horizontal here were protein in grams. So, I actually hid how many grams of protein it was, but this is uniform. So the amount between [points between midpoint of 1st and 2nd bars]. And each one of these is actually only one particular protein amount [points to 1st bar on chart]. So this might be like 5 or it might be 10 grams of protein, but it is just one.



32. P: All right, OK.

33. I: It’s not a span, even though I have it as a histogram.

34. P: Yes.

35. I: OK. Does that make sense?

36. P: Yes.

37. I: And along the vertical is how many boxes of cereal there, that were at that level.

38. P: All right.

39. I: Does that make sense to you?

40. P: Yes.

41. I: Do you think you can interpret that graph?

42. P: Yes.

43. I: OK.

44. I: There were 77 boxes of cereal there. That are depicted there. Could you draw – yeah, feel free [P writes 77 below the graph as shown in figure 2] to write anything down. Could you draw where you think – a vertical line – where you

think the average protein level would be and explain to me how you're deciding where to put it?

45. P: OK. Um, well I see that this level [points to top of bar 3] is the largest and that there's not much above [uses other hand to swipe across bars 4-6] so I can't just pick this [pointing to bar 3] because these 2 [points to bars 1 and 2] overweigh these [points to bars 4-5]. So, I'm going to back it up a bit [brings finger pointing to bar 3 midpoint toward bar 2] and I don't how much to back it up [draws little dash above bar 2, see figure 2].

46. I: It's an estimation right now?

47. P: Yeah.

48. I: OK, so you're saying right about there [points from drawn hash down to top of bar 2]

49. P: Yes.

50. I: Now, let me ask you a question about what you said. You said these two "overweigh" these two [points to bars 1&2, then to bars 5 & 6]. Can you explain to me why that is important when you are finding the arithmetic mean?

51. P: Because the arithmetic mean is almost like the balance point of a teeter totter. And so, I have to find the point at which the weight of these numbers [puts fingers from drawn hash mark to the left of the histogram] equals the weight of these numbers [puts fingers from drawn hash mark to the right of the histogram].

52. I: OK. So what were you looking at? Were you looking solely at the height of the bars to help make that decision?

53. P: I was looking at [craws hands along vertical axis] both the height and the spread of the bars. Because, even though there's not many of these [points to bar 6] it will outweigh [points to bar 5], well it will definitely out weight that. But because these are a higher number [again pointing to bar 6] of protein.

54. I: OK. All right, so you said right about there [points to drawn has mark] and would you do – If I had given you the protein levels, do you think that would have changed where you put it or how you would have figured it out?

55. P: Yes. I probably would have tried to estimate it quickly in my mind, using numbers rather than just the look of it.

56. I: Yeah. When you estimate in your mind, what kind of things would you do? Like, let's say that, let's just – we'll give them numbers [picks up pen and writes numbers shown below histogram in figure 2]. Let's say, we'll give this as 1 gram and it's like coco Puffs up to there, uh 6 and then just with one of them, explain to me what you would do when you estimate in your mind.

57. P: OK. Well, since I'm trying to find the mean of the protein I would say, if this big one [pointing below the 3, bar 3] has 3 proteins [places pen on vertical axis] it

looks like there's about 28 boxes that have 3. So, well, if I could use my calculator.

58. I: I would let you.

59. P: Do you want me to?

60. I: No – no just tell me what you would do.

61. P: I would multiply 3 times 28 [drawing finger from vertical axis that is equal to top of bar over to the top of bar and tracing down to number of protein grams as he speaks], 4 times 10, 5 times [circles fingers above histogram] and I would add and divide by 77.

62. I: OK, so you'd go back to the algorithm and only in this case giving each box its amount, since there's more than one box depicted here [pointing to bar 3], correct?

63. P: Correct [hesitantly]

64. I: All right.

65. P: If I had a calculator

66. I: [laughs with P] I'm not too worried about that.

67. P: And I wasn't in a hurry.

68. I: Right, exactly. All right, let me see if there's any others [referring to interview schedule]. OK. So you had the algorithm and you used this idea of weighting. Now, you called it weighting over here, too [points to grades from first question], with this "assignments". Would this, would finding that blue vertical line [moves hand vertically over the histogram], how is it similar to what you did here? [points to grades] where you found like the 17 above and 17 below.

69. P: Well, uh, it's exactly the same. Because, this is just a picture [points to the histogram] of these numbers [points to the grades]. So, this time we had five scores [referring to grades] and this time [points to histogram] we have 77. But the choices just happen to be narrow. If we had had 77 different ones we couldn't have drawn a picture so quickly, and summarized the data like this.

70. I: All right. I have a third one [Time 9:21] I have a lot of pictures.

71. P: OK

72. I: This one is something like a nurse might see [places figure 3 on table in front of P]. And this is the systolic blood pressure level. The systolic is the top number; I had to go look that up. Um, and what this is, is its nine different systolic numbers that someone might have. And nurses often have to say, well, what's about their average over a period of time. And so they'll look at this chart and they'll find the average. So, how would you do that?

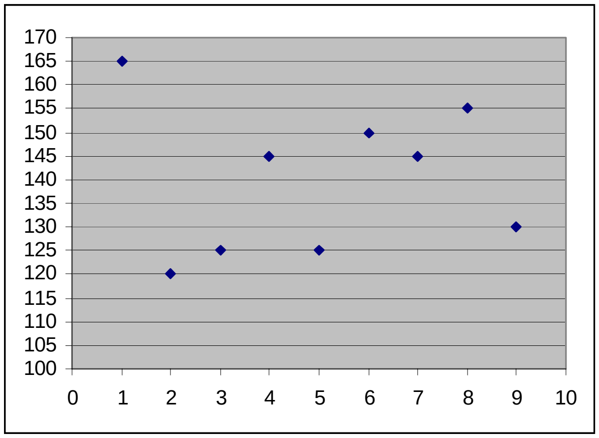


Figure 3

73. P: Is this one person?

74. I: This is one person and it is like taken over nine different times [P is running finger from 1 on horizontal axis up to dot at 165]

75. I: So, here is they're first reading

76. P: OH! Oh, it's a line graph. Oh, all right. So, how would I find their average?

77. I: Yeah, what would their average systolic be about, do you think.

78. P: [turns paper ¼ turn so vertical axis is now the horizontal axis and laughs] I'll turn it this way and . . .

79. I: Oh.

80. P: . . . say, I don't know [as P runs finger along the line at 140] um like . . .

81. I: Why don't you write [P again runs finger along 140], like right along 140 there? [I draws a line over the 140 line in red pen]. Let's just draw a line.

82. P: Yeah, like that.

83. I: OK, so that's estimate right?

84. P: Right.

85. I: And what did you do, what were you thinking about when you made that estimate? What things were you looking at to help you decide it's about there?

86. P: Because the distance between where the point falls [points from dot 1 to the red line, then points to each dot above the red line] and – is that what you're going to drive at, is like the least squares regression line?

87. I: Not really [both laugh]. Actually, I'm kind of interested in how you're estimating this and at the end of it, I'll tell you what I was really looking at.

88. P: OK. Because I want to find the one line that, the one [draws hand along red line] that has the least [draws a vertical line from each dot to the red line, approximately perpendicular, see figure 4], well actually, that this side [points to

dots above the red line] is exactly the same distance as this side [points to dots below the red line].

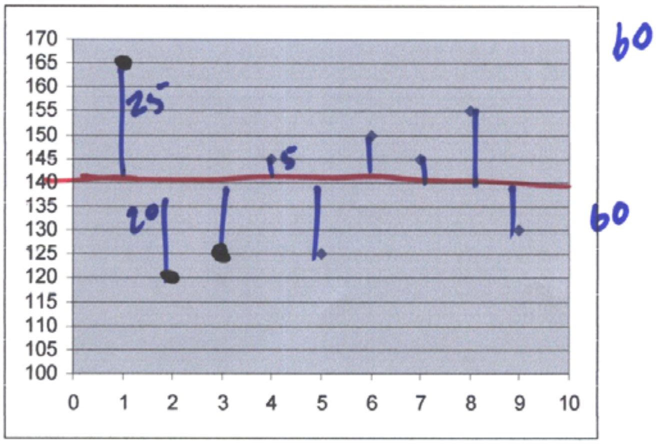


Figure 4

89. I: OK. So, if you were, now that you've drawn this, if you were to make it more exact, what would you do with these lines here to make sure that your line was a little more exact?

90. P: I would square them, average them and unsquare them.

91. I: OK. Is there anything you can do with eyeing it up, with what I have there? And say, like, "yeah, I think that's a pretty good estimate." By just looking at what you have there?

92. P: Oh. I could measure the distance. So 25 [writes this on line from dot 1 to red line, figure 4] and 5 [dot 4] and add up these distances [runs finger along dots above the red line] and measure theses [runs finger along dots below the red line] and add them up. And if they are the same [points to above and below the red line], then I'm good to go.

93. I: All right. So let me do, real quick, this is 30, 40, 45, uh 60 about? Does that look like – I'm adding up as I go, 25 and 5 is 30, so that's about 60.

94. P: 45, Yes. 60. [writes this to the right of the graph above red line]

95. I: This is 30, 35, 40, 50 [counting lines below, laughs] 60.

96. P: [laughs and writes 60 to the right of the graph and below the red line]. Surprise. Surprise.

97. I: So, uh, how'd you do? On finding the average blood pressure, there, given the numbers we just came up with? How'd you do?

98. P: Perfectly

99. I: Yeah, pretty good, huh? [laughs] See, I told you these tasks weren't that bad. So, basically, what you've done is, again, like you did here, right? [holds out grades from 1st question]. Only you were estimating, using your eyes to figure that out.

100.P: Yes.

101.I: OK. Let me, I'm going to use this black just to highlight two dots here. [highlights dots 1 and 2], figure 4]. You've got this guy here and this guy here.

102.P: Yes.

103.I: If I wanted you to find the average [takes out a blank page and covers all dots accept 1 and 2] of just those two, what would you have done?

104.P: I would have added them up and divided by 2

105.I: OK, so you would have taken the 165 and [P writes 20 in on line from dot 2 to red line] and the 120, right? Which is there [points to dot 2]

106.P: Uh, OH!

107.I: Is that what you're saying?

108.P: No, I was thinking [runs finger along 140 horizontal line], because we were, I was thinking of looking from the distance [traces line from red line to dot 1] as my estimate. But I guess, yes, if you had just two points it would be faster to add and average – oh, you can't. Let's see [points to dot 2]. Then you'd have to add 160 [sic], 165 and 120, yes.

109.I: OK, when you said "Oh you can't", were you looking at the 20 and the 25?

110.P: Yes, yes.

111.I: And what was the "oh, you can't"?

112.P: Well, because that was from my estimate [points to the red line] so if I want to find the true mean, I better stick to my [points to dot 1 and dot2], to the true values, not . . .

113.I: Oh, I see what you're saying. Not the estimated one.

114.P: Right

115.I: OK, so when I add this third point in here an 125 [moves paper so only dots 1-3 are seen and circles dot 3 with black pen] right here, um. Well, let me go back to these two here [hides dot 3 with paper and points to dots 1 and 2]. So, when you did this, and you add these two up and you divide by w it ends up being like, right [places a dot on the vertical axis at about 142] here someplace.

116.P: One . . . yeah

117.I: 142 something, right?

118.P: Right.

119.I: All right, now when I add the third one, okay [moves paper to reveal dot 3], how does that change your arithmetic mean, and why does it change that?

120.P: I added a third point that was on the low end [points to dot 3] so it's going to drop the mean.

121.I: Why is it going to drop it?

122.P: Because now we’ve got; in order to keep this balanced [points to three dots] we’ve got two low scores and one high score so the average is going to be lower now [runs pen along red line] then it used to be.

123.I: OK. All right [Time 14:52] We’re doing well. You’re doing a great job here, thank you. Next one is a, a problem I got from a friend in math ed. So, it’s a little bit challenging, but it’s a fun one [places figure 5 on table in front of P].

A class of students took a test. The class average on the test was 68. The average grade of the students who passed was 80 and the average grade of the students who failed was 64. What percentage of the class passed?

Figure 5

124.P: OK

125.I: I’ll let you read that, and basically, if you have any questions about it.

126.P: OK

127.I: All right, do you have any questions about the situation, are you able to understand what’s going on?

128.P: Yeah, I can’t think under pressure.

129.I: That’s OK. Because, what I am going to ask you is, um, let’s suppose that I take away your power of using a variable. OK? So you’re not allowed to say passing is p and failing is f.

130.P: Oh, right.

131.I: OK, so you’re not going to use variables. But, let me ask you, how might you go about solving that? What kind of things might you do to try and solve that problem?

132.P: All right, uh, let’s say, if everyone’s was 68 [draws a circle and writes 68 inside, see figure 6], then my [draws two smaller circles to the right] uh, this is 80 [writes in the top circle], this is 64 [writes in bottom circle and reads from paper] “what percentage of the class passed?” I would pretend that 68 is the median and I would see what happened, uh. This is a good example of why measures of central tendency are insufficient and you need a measure of variability.

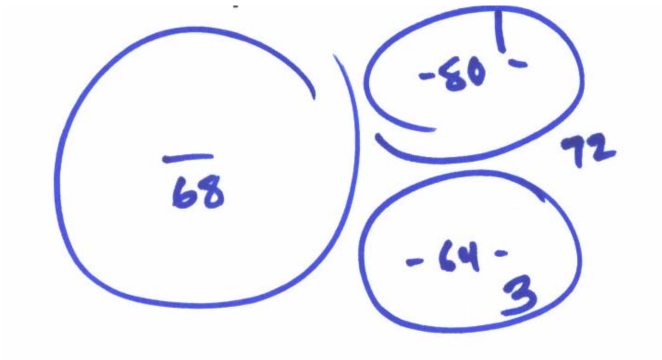


Figure 6

133.I: [laughs with P about comment]

134.P: Let's see here.

135.I: Let me ask you this: is there, you said you were going to go to the median. OK, when I think about median, I don't think about variability. Does the median take variability into . . .

136.P: No.

137.I: OK and so, you were actually going to go with, okay, I'm going to go with, say I take the median, then what were you going to look for next?

138.P: I was going to set up, sort of, this hypothetical situation [runs fingers around all three circles] and examine it in my mind to see what happened.

139.I: OK. And as you examine it what kind of conclusions are you drawing as you look at that? That that is helping you conclude?

140.P: [3 second pause] OK. Can I actually answer this question? The class average was 68 [going from reading of question down to circles drawn with pen, pauses about 7 seconds]. The average of those who passed – what percentage passed? With the average of 68 and I've got two students. One passed and one failed, the average isn't going to – OK, I'm going to [picks up calculator]

141.I: Feel free to. Would you just tell me what you are punching in so I know?

142.P: Oh. 80 plus 68 equals divided by 2 equals, OK. They are going to force the mean to be 72 [writes this to the right of the two circles]. Since I know that the mean is lower than that, I have to have at least two people down here [points to lower circle with 64 in it] one of – [3 second pause] OH! I guess I can solve it.

143.I: What made you say that? What were you thinking when you all of the sudden went "OH"?

144.P: Let's see. Brother, yeah, now I need algebra. Let's see. Well, because it doesn't really matter [laughs and moves paper around] how to say this. Let's see. I have to force the size of this one [points to the circle with 64 in it] the proportion of this [points between two smaller circles] of the sizes, to force the mean to be 68 [pointing to both the 72 and the 68]. So I can control the percentage, even though I can't control the number.

145.I: OK. All right. So, when you are controlling the percentage, what are you going to do to these two [points to both smaller circles] numbers? Like you started and said "if I have one and one it would be 72", then you said, "well, if I make this 2" [pointing to circle with 64 in it] and one [points to circle with 80]. Is that what you were thinking? You didn't say "and one".

146.P: Yes.

147.I: OK. So you would play around with the proportion of this to this [points first to 64 circle, then to 80 circle], until . . .

148.P Until I hit 68.

149.I: Until you hit 68. and then that would give you the percentage. That's basically how . . . ?

150.P: Right.

151.I: And then, when you way you would play around; would you do it on your calculator? Like do 64 plus 64 plus 80 and divide by . . .

152.P: [speaking over I] Yes.

153.I: . . .three and see how that landed you.

154.P: Yes.

155.I: Go ahead and try it. Let's see where it lands you. Since that's what you would do next.

156.P: OK, so now I've got three students, let's say, 80, 64, 64 [writes these down figure 7, then uses calculator to add these] equals 208 divided by 3 equals 69.

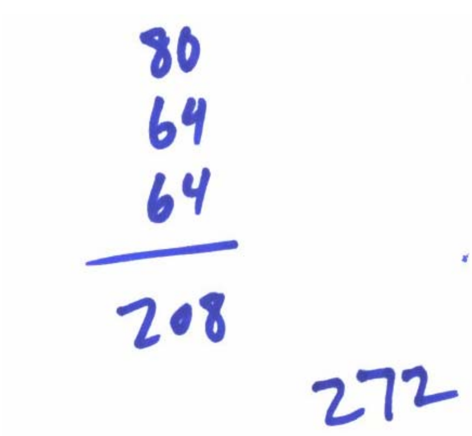


Figure 7

157.I: Still not there, but,

158.P: Right. OK [working on calculator] 208 plus 64 equals 272 divided by 4 [writes this number down]. Oh good, I don't have to do this too much more.

159.I: and that equaled 68.

160.P: 68. Okay, so we've got one to three [writes 1 in 80 circle and 3 in 64 circle] so 25 percent passed.

161.I: Great. And so, basically, you were playing around with adding more until you got a proportion that landed you here [puts hand over 68] and so my class could be 20 students, but I know that only 25% of them passed. Or my class could be 40 students.

162.P: In order to keep the mean there [pointing to all three circles] it would – yes.

163.I: Is – do you see what you did there [points to circles] as related to any of what you've done before [holds out previous three sheets]? In any of these, do you see, what that –

164.P: Yes.

165.I: How is that related?

166.P: It still has to do with the weight, and like the teeter totter effect that I was trying to figure out how much weight to add here [points to 64 circle, then to 68] in order to get the balance point to be here.

167.I: OK. I have a really big question [Time 21:42] and I'm going to look at my paper to make sure that I ask it right.

168.P: OK

169.I: Because this is kind of an interesting question that keeps coming up in my interviews. And that is, when I first asked you to find the arithmetic mean you said, "I'm going to add everything up and I'm going to divide by how many there are." Basically, and if I looked up in a mathematical dictionary what arithmetic mean is, that's the definition they give. They say, sum up all of your data and then divide by how many data points you have.

170.P: They toll you to calculate it, right? They don't give you the definition.

171.I: They do. However, in this interview, for every single one of these problems that we've worked on you have referred to it as a mathematical point of balance, in a sense. In this one [points to histogram in figure 2] you said it's a place where, like a teeter totter it balances . . .

172.P: Right, right.

173.I: And you're using weight on one side and weight on the other side.

174.P: Right

175.I: Like keeping how far away from each other the same.

176.P: Yeah.

177.I: and my question is, how do you determine that the arithmetic mean is a mathematical point of balance from this standard algorithm that we’re so used to using; adding everything up?. . .

178.P: Oh.

179.I: How do you determine that it is a point of balance?

180.P: [takes sheet of grades from first problem and places on table] Because, if – let’s just figure out what the mean is, so [uses calculator to find the mean of the five grades] so when I make the theoretical point it matches.

181.I: OK that works for me.

182.P: OK, so the sum is 425 [writes below list of grades, figure 8]. Divide by 5, OK, 85.

183.I: Right

184.P: OK, so this is the mean [writes $\bar{x}$ = next to 85 in list, figure 8] Um [pauses] because we have calculated the mean, we know this is the center point [points to 85] but how we can use that is because we know that if we say, the distance between this [draws a line from 85 to 97 and labels as 12, figure 8] is 12. And the distance between = plus twelve [adds a + next to 12]. And the distance to this [draws line up to the 90 and labels +5] is plus 5. we’ve got [writes an 85 below grade list, draws a horizontal line above and then writes 17 to the right] 17 on this side [adds a carat mark just below the 85, no comment with this]. And we know that we’ve got [points between 80 and 85] negative 5 here [writes -5 between two numbers] and negative 12 [points between 85 and 73 and writes -12 between next two numbers] so we’ve got negative 17 [writes on left hand side of the horizontal line]. So that’s why the weight is the same.

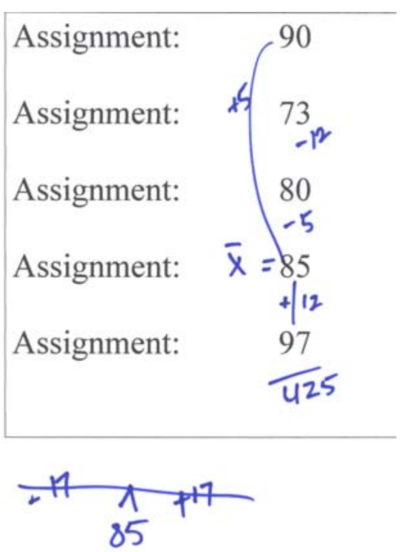


Figure 8

185.I: OK. In your experience with the arithmetic mean, have you ever been asked to figure out why it is a mathematical point of balance? [Time 24:18] that you can think of? I mean has that become an important use, ever?

186.P: [7 second pause] Well, I teach statistics, so that's where I'm really trying to think. Do you mean, as I teach statistics, have the students raised the issue? Or do you mean as I do . . .

187.I: Has it ever [over P], yeah, as you do statistics in your work because I know the students are probably going to bring it up.

188.P: OK. Um [pauses 3 seconds] balance point [6 second pause]. No, I don't think so.

189.I: So, in your statistical work, mostly it is sum everything up, divide . . .

190.P: Well, I use a computer, so . . .

191.I: Yeah, it does the summing and dividing for you

192.P: Right. I just get the mean and then interpret it.

193.I: So when you get the mean and you interpret the mean is there, in the interpretation part, is there some part that's related to balance do you think?

194.P: All right, let's see, um,

195.I: I was going to say, is your brain running through all the possibilities?

196.P: Yeah, I, right. Well, because I keep bringing it back to my own examples and in my work I face a dilemma because often I'm running means on Lichert scales which we know you're not really supposed to do.

197.I: Right, yeah.

198.P: But, in practicality you have to do. So, do I think of it as a balance point? No, I just think of it as one way to represent the typical number.

199.I: OK. All right. Good. Thanks. [Time 26:13] All right, I have another little picture here for you. So, we'll move some of these so you can have a nice clean space [places sheet, figure 9, on table after moving papers]. And that is, this is: we're going to pretend.

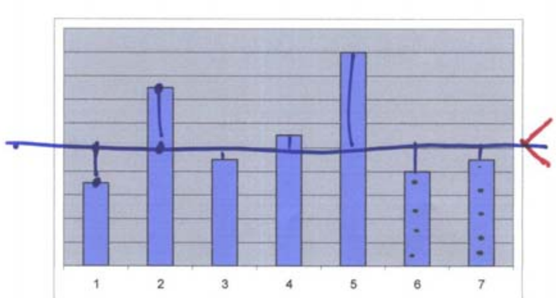


Figure 9

200.P: OK

201.I: That these are buildings. That this is not a bar graph.

202.P: [laughs] OK.

203.I: But these are buildings. And these are buildings one through building seven [points to each bar]. And, if I were to ask you to find the average height of the buildings, what might you do to find the average height of these buildings?

204.P: You mean of my distribution of buildings?

205.I: Yes, your distribution of buildings [both P and I laugh]. Of those seven buildings

206.P; Huh, I guess I'd, hmm, I'm trying to think of them as buildings and not as, as . . .

207.I: Yeah, it's hard to do when you're so used to the same bar graphs – as something else.

208.P: I'd say, alright, Well, this is the middle building [points to building 4 on horizontal axis and laughs]. I would say alright, if I were to pick this I've got four [points to buildings 1, 3, 6, and 7] lower buildings and two [points to 2 and 5] higher, so that's no good.

209.I: OK

210.P: If I were to pick this [puts a finger on 2 and one 7], oh shoot, two buildings are the same, so then I've got two lower [points to 1 and 6] and three higher [points to 2, 4, and 5]. Yeah, three highers and two lowers. I'd say that it has to be somewhere between [points to building 3 and building 4] this building and this building.

211.I: Are you looking at it this way? [places side of hand in a vertical position] like putting it here [puts finger on a point on the horizontal axis between building 3 and 4] or are you looking at it this way [puts hand in a horizontal position and slides up the graph]?

212.P: No, I'm looking at it this way [runs finger up to indicate the second way, a height]. Height.

213.I: Yeah. OK. So can you show me a horizontal line where you think it might be then?

214.P: [picks up pen and turns the paper ¼ turn and draws a horizontal line as seen in figure 9].

215.I: Since I'm trying to figure out, I had someone else do it the other way, that's why I asked.

216.P: Oh

217.I: OK. So you are saying about that line [points to line P just drew] is about the height of the building.

218.P: Yes.

219.I: Now, that's an estimate. And you did it by looking at buildings over it and buildings under it. Right? Now is there any way that you could make the more accurate using what you've drawn? Now that you've drawn the horizontal line.

220.P: Yes. The same thing we did on the systolic chart. Um, I could [runs pen up and down left side of chart] where's my axis?

221.I: Yeah, there is no axis. But, I did draw lines and you can call them units if you'd like.

222.P: OK [draws a vertical line from his proposed mean to the top of building 1]. So, I would just [continues and draws vertical lines inside the buildings that are higher than proposed mean to the mean and from the proposed mean to the tops of buildings that are shorter] compare the distance to the mean – to my estimate [points to each drawn line] and find the lengths of the line and hope that this side matches [points above proposed mean and then below the proposed mean] with this side.

223.I: and, let's say that for some reason you're a little off and you have more up here [points to above the proposed mean]

224.P: All right

225.I: Where would the line, where would the mean go?

226.P: If my estimate, you mean?

227.I: Yeah, let's say that your estimate of over is more than your estimate of under at this point. I think you're pretty close, but, you know, if it were?

228.P: If I'd, that would mean I'd have drawn my estimate [runs pen along horizontal proposed mean] to low, so the mean would be higher.

229.I: OK. Now, is there, let me just tip it this way [turns sheet ¼ turn]. All right. Do you think that you could call – let's say that you were accurate [picks up pen and puts red carat mark on side – see figure 10], we'll assume, is there any way that you could call that your point of balance? [4 second pause] Do you see that as being a point of balance at all?

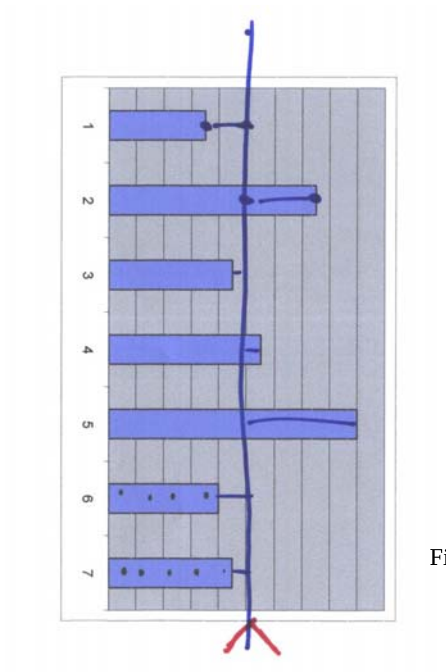


Figure 10

230.P: Yes

231.I: How?

232.P: Because the distance from that point, um, because the buildings that are higher than that point equal the buildings, the distance of the height of the buildings to that point.

233.I: OK. See as a 6th grader, if I were a 6th grader, I would have a hard time with this, right? Because a 6th grader, is going to look and say I've got, let's say one, two, three, four, and a half, right [puts dots on each unit inside bar 7]. And one, two, three; I'm counting the units here.

234.P: Yeah, yeah.

235.I: four [putting dots inside bar 6]. So that's like, so that's . . .

236.P: Right

237.I: And they might add up and [moves hand on the lower part of every building], say, well all the weight on this side, if you look at the blue on this side of, the blue bar on this side is an awful lot more than the blue bar on this [now points to only the parts that are above the estimated mean line]. Do you see what I'm saying?

238.P: Oh . . .

239.I: And so that would be hard for me as a 6th grader to see that as a point of balance. But you explained it differently?

240.P: Oh, that's because I'm not looking at from the base [points to very bottom of bar 1] oh

241.I: hang on a second, you can keep going [I switches audio tape to second side while talking], I'm listening

242.P: OK

243.I: You're not looking at it from the base.

244.P: Yes. I mean, you're not calculating the balance point from the base, you're calculating the distance from the top of the building to the balance point [redraws on building 1, line from top to proposed mean]. So, . . .

245.I: Oh, OK. So, I see, you're looking at the top of the building to the balance point, on either side, no matter what side it's on?

246.P: Right.

247.I: And this is lacking [points to building 1] and this is over [points to building 2], is that right?

248.P: Yes, that is correct.

249.I: All right, so how is this related to the previous stuff [points to building picture and then stack of other papers]? Do you see it as related to how you did the other things?

250.P: Exactly the same.

251.I: Yeah. You know the hardest; this is the question that I just have not been able to figure out how to get an answer for, that makes sense. And I'm going to draw [takes clean sheet of paper and pen, figure 11] a picture, because we're actually going off the interview schedule a little here. But that is, if I have points on a line and I say this is my arithmetic mean [first draws horizontal line and draws carat in middle below], my arithmetic mean is here because it balances here. And maybe I'll even use the numbers that we used before, which is 97 [begins to mark on horizontal line as he speaks] 90, over 85 was my balance point, and then I had 80 and 73, I think it was.

252.P: Right.

253.I: I still don't see how when I take 73 [begins to write column addition as he speaks] 80, 85, 90, and 97 and I add those up and get whatever it was, 200 and something, what was it [checks previous work] 425 [writes below column]. I get 425 and then I take 425 and divide it by 5 to get my 85 [writes division problem to the left]. I'm still not sure how this algorithm . . .

254.P: produces that.

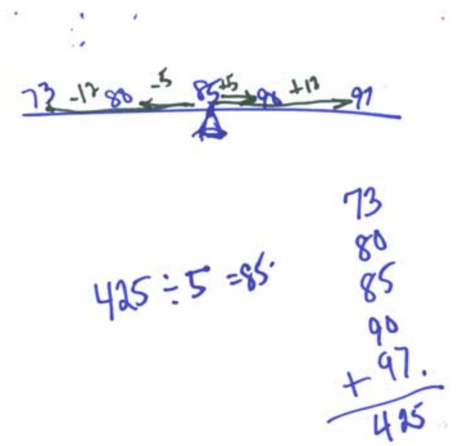


Figure 11

255.I: Yeah, can reduce to this. Because when I reduce to this [picks up black pen and begins to add to line] I'm looking at, you're looking at, at least each time, you've been looking at this is five [draws arrow from 85 to 90 and labels 5, adds plus sign], plus five. And this is plus 12 [draws arrow from 85 to 97 and labels]. Right?

256.P: Right.

257.I: And this is minus five [draws arrow from 85 to 80 and labels] and minus 12 [draws arrow from 85 to 73 and labels]. And you're looking at that. And I'm not really sure how this [puts hand over lower half calculations] is related to that [puts hand over the number line]. Truly, I don't know.

258.P: Yeah, It's magic [both laugh]

259.I: It is, OK. "math-magic". Is there anything that you can see there that you would say, "I think it's related because of this . . . "

260.P: Um, I don't know. Let's see. If I thought of these as weights instead of scores, like I have a 73 pound student and an 80 pound student

261.I: OK, yeah.

262.P: Um [talking very slowly and drawn out] then, these, in order for this [points to 85] to be the middle point I have to come up with two students [points to left side of number line] to weigh exactly the same as [now points to right side of number line] these two other students. Is that enough?

263.I: Would that do it? So, uh, I don't know. This is just, you know, to answer your question, no, I wasn't trying to get at one particular way. What I'm trying to understand is how we come up with this idea of balance because we use it all the time. In our use of the arithmetic mean we use balance a lot. And yet, when you look at how we learn [pointing to calculations] arithmetic mean, we learn it this way. And so I'm really curious how we get from learning this to understanding it as balance.

264.P: Well, this is like the shortcut [runs finger down calculations] to, and one you get the number, then you look at [runs finger along number line].

265.I: So in a sense, do you look at this [circles number line with finger] as more representative of what the arithmetic mean really is? And this is sort of the short cut to get to it [points to calculations]? Is that what you're doing?

266.P: Yes, I suppose.

C.2 Telephone Interview Transcripts

C.2.1 Telephone Interview with C

1. I: First of all, let me review with you what has happened with the data and everything about it. It's been almost two years since I interviewed you.
2. P: That's hard to believe
3. I: I know. But in the interview, I think you might remember, I was asking you, giving you tasks that were related to arithmetic mean and I have been working on analyzing the data that I collected. And it's been fun, sort of. But I came to the conclusion that, for you, there were particularly two ways that you thought about the arithmetic mean. The first way was related to the algorithm. OK? And that was – at least the operations related to the algorithm. In other words you did things that were related to adding up all the values and then dividing by the number of data points that you had. And then the second way that you thought about arithmetic mean was as if it were a mathematical point of balance. And that is, you found a place in the data as if you were balancing out all of the values in the data. Does that sound like a fair description of what you remember?
4. P: Sure
5. I: Sure?
6. P: You know you probably know what I remember better than I do.
7. I: Well, I probably do! [laughs] But I wanted to make sure – if you think about your own understanding of arithmetic mean you think that would be a fairly – you could agree with those two ways.
8. P: Yes.
9. I: Maybe not limited to those two ways, but that would be two ways you could . . .
10. P: Of course.
11. I: OK. So, I'm going to ask you a few questions specifically about those two ways. Which is the algorithm and balance. OK?
    P: OK.

12. I: Can you explain to me how you understand that the arithmetic mean defined by the standard algorithm and the arithmetic mean as a mathematical point of balance are the same thing? [5 second pause] Understand the question?
13. P: Yes I do.
14. I: OK
15. P: You don't mind giving me a minute or two to collect my thoughts?
16. I: No. Collect your thoughts, but collect out loud.
17. P: I forget about collecting out loud.
18. I: Yes, you've got to collect out loud for me.
19. P: Um, well, how is the algorithm —how are those the same?
    I: yes.
20. P: Well, I guess the first thought that comes to mind is the idea of the standard algorithm is – its kind of a fair, its kind of a sharing. You know you're totaling up all that the – the total quantity and you're dividing it by the number of slots, in a sense, you would have to distribute that quantity. And so, in that sense, its kind of a – I think of it visually in terms of leveling.
21. I: OK.
22. P: And so, I guess as I think about point of balance – hm. Its – that feels different but also related in the sense that—I don't know if I can make the same connection to the leveling off with the point of balance. But I'm drawing some pictures here.
23. I: OK.
24. P: One – uh maybe I'll need to fax these to you.
25. I: You can always fax them or you can either, that or you can describe to me what you're doing right now.
26. P: Well, I'm thinking visually in terms of when I was talking about the algorithm I was – I made dashes, like five dashes, kind of like hangman spaces, you know the game hangman?
27. I: Yeah – yes.
28. P: And thinking of, OK, We are adding up the total quantity. I don't have a better term for that. And dividing it equally among five different spots. So you can think of that maybe as bars of a bar graph and you are getting them all to the same level.
29. I: OK
30. P: And in terms of the balance point – that's – I think there – I only have a loose sense of that connection. And that would be – and I have a little balance beam drawn here.
31. I: OK

32. P: And you know, if I put five dots on here, randomly. Two on one side and, two to the left of the fulcrum and two to the, three to the right of the fulcrum. The balancing – I mean the idea of leveling is I guess, moving them all to the – well, let me take away the fulcrum because that makes it harder if you're trying to move it to a preconceived spot. So have a line – balance beam, no fulcrum yet. And, I'm moving these dots that are randomly spaced along the line to a point. And that point would be where the fulcrum should be placed.

33. I: OK. How are you choosing where to move them?

34. P: Well, it's going to have to be in terms of what I'm – the distance I'm moving one dot on the right would have to be equal to the total distance that I move dots on the left.

35. I: OK. All right.

36. P: Or the – and so I would keep moving. In a sense, I think, all right, let's say these are all equal – equally weighted dots. If I move a dot one unit on the left, then I've got to move a dot one unit on the right. And so I could continue to move dots on each side until I got all of them to the same point – all stacked up at the same point.

37. I: And so that's like leveling in what way?

38. P: That's like leveling in the sense of when I think of calculating an average, an arithmetic mean, I will take values away from one number. Take an amount away from a number and add it to another number. And I'm looking for that point that the two, or that all the numbers would be the same value.

39. I: All right

40. P: So if you read off like a list of grades – lets say they were 88, 86, 82, 78, 76.

41. I: OK.

42. P: And I was going to calculate an average. Then I could kind of ball park some kind of midpoint as my target. So let's say, 82! [he sounded surprised that he had such a nice set of numbers because he paused before choosing 82].Which actually looks to be exactly what the average is. Then I would say well 88 minus 82 that's minus 6. So, I've got to add 6 to another number. So, I would have that my 88 is now an 82, my 76 would now be an 82, I would have an 82 and my other two numbers are 86 and 78. So to make 86 82 I've got to go down four. And then go up four from 78.

43. I: All right . . .

44. P: So, in that sense, I see those two processes as identical, just a different representation of the same process. In terms of moving the dots to that same point and calculating an average the way that I did.

45. I: mm-hmm

46. P: Now, in terms of the connection to the standard algorithm, it might be easier to connect my numerical leveling to the standard algorithm rather than the visual that I was trying to describe a minute ago. So, that numerical leveling process I just

described [pauses] I wonder if it has something to do with division as subtraction? But that's a thought for another day.

47. I: So you are thinking of – the algorithm calls for division
48. P: Yes.
49. I: So you're thinking, OK, what is the, what's the operation of division give you – I mean, it led you to subtraction?
50. P: Yeah, well I'm just – you see this is a, kind of a – we're turning the process into – calculating the arithmetic mean into a series of additions and subtractions. And uh [pauses]
51. I: And the algorithm is addition and division.
52. P: Right. I don't have a clear way to articulate that connection, but
53. I: Ok. So, let me ask you another question. If I'm just looking at the standard algorithm, is there anything in that algorithm that would suggest balance to you?
54. P: Yeah – the division.
55. I: OK. Why would division suggest balance?
56. P: Uh – because you are partitioning a number into – a total into x number of equal parts. In my case, five equal parts.
57. I: And what about that makes you think about balance?
58. P: I think that if you go back to my description of moving the dots to a balance point. It's kind of the – it doesn't immediately make me think of balance. But, if I had to construct an argument, it would be something along the lines of, OK: division – the standard algorithm is partitioning the total into five equal parts, right?
59. I: OK
60. P: That would – I guess it is more – I think leveling is more of the connecting point for me.
61. I: And it is the connecting point in the way you described earlier?
62. P: Yes.
63. I: OK I won't make you rehash that then. Let me just tell you that the conundrum that I've been dealing with a little bit is that we seem to start to learn arithmetic mean as the result of an algorithm and students tend to start with the algorithm. And the question that I'm asking is how do we develop this into an understanding of arithmetic mean as a point of balance? If you thought about that, or even if you thought about it in your own history of understanding arithmetic mean; do you have any idea as to how it develops into an understanding as a point of balance?
64. P: [7 second pause] No.
65. I: [laughs] Good answer. Ok, well that's pretty –

66. P: It is almost like those two – and I don't know if it's like this with other people – but I think the difficulty with which, the difficulty I have specifically with making that connection. I feel like there's still a gap. And the difficulty with which that connection – it is not something that's conscious, that I'm consciously aware of; makes me initially think that those are two very different understandings of arithmetic mean. And I could come up with some kind of loose connection but without a lot of thought, I'm not sure of how to think of that connection. And maybe it is two things that you have to develop separately rather than trying to make the explicit connection or go from one or the other.
67. I: Well, that's pretty much the questions I had to ask . . . .

C.2.2 Telephone Interview with L

1. I: Really, let me tell you what we're doing, what I'm doing. You remember about a year and a half ago I interviewed you, right?
2. P: Uh huh
3. I: And I was working with the arithmetic mean and I've been working on analyzing what I can learn from the data I collected.
4. P: Uh huh
5. I: and as I've worked I've come to realize that there are two different ways that you worked with the arithmetic mean. Ok, one way that you worked with it was with the algorithm, or at least with the operations of the algorithm. Like you added things up and you found the value of all the data points together and you divided by how many data points you have . . .
6. P: Uh huh
7. I: The second way that you did your work with the arithmetic mean was as a mathematical point of balance. You know, that you found a place in the data that was as if you were balancing out all the data values. Does that sound like a fair description of what you remember from the interview? I mean it has been a while, but . . .
8. P: yes, yes, it sounds like a fair description, yes.
9. I: OK, good, so
10. P: they both could be quantitative though
11. I: Yes, both of them could. So that is actually what I want to talk to you about. And I'm going to ask you a few questions about the two approaches.
12. P: OK.
13. I: OK? The first question is this, can you explain how you understand that the arithmetic mean defined by the standard algorithm and the mean as a mathematical point of balance are the same thing?

14. P: Um, the – assuming that they are the same number of data points so that we are comparing apples to apples . . .
15. I: Uh huh
16. P: the uh, the true, the algorithm for the mean is very clear cut, you just add all the numbers together divided by the number of them, you get the true average. Um, what you get is equivalent to the second way, you just find the difference, you look at the difference based on the data points you have you find the point of balance. I mean there is a point representing the true average.
17. I: Yeah.
18. P: But to find the point – both would be the same as one another but in the second way how to find the particular point could be a challenge. It can also make life easier; just by because you put everything on the graph. You have so many different points and you just, judging by inspection you can judge the difference – to find the balance point of all. Then you can check the – by comparing this particular balance point with all the data you have.
19. I: OK Yeah, is there anything that in the algorithm itself. In the add up and divide that suggests that that will be the point of balance?
20. P: Hmm. I did not understand your question immediately.
21. I: Yeah. OK. Let me see if I can make it clearer, OK?
22. P: uh huh
23. I: When I look at the algorithm, it says you're adding everything so that's like accumulating an amount, a value all together, right?
24. P: Uh huh
25. I: And then I divide it so I kind of share it evenly across . . .
26. P: Yeah.
27. I: . . . but is there anything about adding and dividing or that algorithm that would even suggest that that should be a mathematical point of balance?
28. P: Yeah. If you put all the (not sure what he said here) on two different sides of the equation. OK? On the left hand side you have all these data points being added together and divided by the total number. On the right hand side you have the average value.
29. I: OK
30. P: Now, if you arrange the equation and you multiply both sides by the total number of n of the average value
31. I: Yeah
32. P: Then you put them, on one side you have the average value minus each individual data point and the summation must equal zero.

33. I: Yeah.

34. P: The point of balance must be chosen just so that it can be the true average.

35. I: Could you say that one more time? That last sentence you said?

36. P: Yeah. The – beginning with the equations – you have the summation of all the data points divided by the total number of data points so you are looking at the true average.

37. I: Right

38. P: Now if you multiply both sides by the total number, the number of data points.

39. I: Right

40. P: And then you move the left hand side of the equation which represents the summation of those n data points, the right side. Then you have the, you would take the n different terms. Each point, the difference between the average and the individual data points.

41. I: OK

42. P: And then the summation must be zero.

43. I: OK.

44. P: That means the point of balance must be the true average such that the summation of the, all the individual points would be zero.

45. I: OK. I think I understand and what you are saying is when you get those differences, the deviations from the mean, that you are summing those and it is becoming zero.

46. P: Becoming zero, yeah.

47. I: And that's the same as a point of balance?

48. P: Yeah and [lots of hesitation here] a good way to know what mathematically the proper balance means.

49. I: Yeah

50. P: And you see this mathematical mean is different from the standard deviation which is summing the difference between the average and the individual points. You raise the difference to a square and then you add them up and take the square root.

51. I: Yeah. Right

52. P: That gives you the standard deviation which measures how much scattering the data are from the average value. This point of balance represents the difference between the true average and the different points.

53. I: Yeah. OK. You know the thing that I am trying to figure out and this is really what is holding up my dissertation right now is you know, it's not real clear to me how we learn – you know we usually learn first the algorithm, for the arithmetic

mean: add everything up and divide. And yet, all of my experts that I've interviewed understand arithmetic mean as if it is a mathematical point of balance. And I'm not real sure how we develop that. You know, how we get from the algorithm so that we understand that it is also a mathematical point of balance. We know it is. But I'm not real sure how we develop that. Do you have any clues?

54. P: It seems we define a reference point in terms of the – everything refers to the mathematical average; using the average as the reference point.

55. I: Yeah?

56. P: (something) using the difference from the value of each individual data point will be either positive or negative to get the point of balance, the average value. And then the point of

C.2.3 Telephone Interview with J

1. I: Um, I don't know if you remember but about a year and a half ago I interviewed you?

2. P: I do remember but I can't remember the problems you gave.

3. I: Oh don't worry, that's fine. I didn't expect you to. I didn't expect you to remember them at all. But I've been working on analyzing the data I collected and I've learned some interesting things, but I have a question that came out of it that I really need to follow up and ask. So, let me give you a little background so you know where we are coming from and hopefully get your mind thinking back the way it was then. And then I'll ask my question. I was working with studying the arithmetic mean and how you understood it. And as I was looking at the data there's really two ways that you were able to describe how you worked with arithmetic mean. The first way was strictly the algorithm; that you add up everything and divide by how many data points are there.

4. P: Right.

5. I: And you did that in a number of different ways but basically it was adding and dividing. So, I've been calling that an algorithmic way of understanding. The second way was that you understood arithmetic mean as if it was a mathematical point of balance.

6. P: OK

7. I: All right. Now does that sound like a fair description to you of understanding?

8. P: Yes [while interviewer still talking]

9. I: It does?

10. P: Yes.

11. I: OK. So I have a question about these two approaches.

12. P: OK

13. I: Could you explain how you understand that the arithmetic mean defined by the standard algorithm and the mean as a mathematical point of balance are the same thing?
14. P: Oh, why I see it as the same thing?
15. I: Yes.
16. P: Hmm, I'm not sure. I just – it is almost as if those two ideas come from very different sources. The point of balance idea comes from the idea of a centroid, I believe. Like a teeter-totter.
17. I: Yeah.
18. P: And the fact that that centroid is just a mean probably requires some sort of mathematical proof. And so, I don't, just off hand, can think of a way that I could argue that that's the case. I'd have to think about that. Because one of them is a mechanical/physical
19. I: Yes.
20. P: . . .idea and the other is algorithmic. And I don't think there is any obvious way to connect between the two.
21. I: OK. So, what has really been the interesting question for me is as I'm looking at this is how do we develop that it is a mathematical point of balance?
22. P: Ah.
23. I: You know we start with the algorithm, so how do we get there?
24. P: Right.
25. I: Yeah – do you have any ideas? [laughs] off the top of your head.
26. P: Off the top of my head, I suppose it is the teeter-totter idea. You put a large – say you want to think of the observations as masses or weights. And, the further out the bigger the mass. So, as we all know, if you put some if you put somebody way out at the end of the board, then you can balance it out with someone closer in on the other side if they weigh enough. Um, you know, I'm not sure. Beyond that I really hadn't thought of it too much. I don't use that in the classroom.
27. I: Yes. You basically use the algorithm when you teach?
28. P: Exactly. Some textbooks try to use this balance idea, but I don't think it does anything but give the student an alternative way to view the mean. At least, I believe that's the case.
29. I: Yes. Have you ever looked at arithmetic mean, the algorithm, and instead of calling it balance, you look at the deviations, that sum equal to zero?
30. P: Yes, that works.
31. I: Yeah, it works. But when I think about that, is there in the algorithm itself that would suggest that that would be true?

32. P: No. Again, Not that I would see of the top, no.
33. I: OK.
34. P: We use the fact the sum of the observations minus the mean is equal to zero a lot of places in statistics. But then that just gets into more advanced areas and you usually have to point this out to the students. It is not that obvious that that is the case. That it just is.
35. I: All right, well that was pretty much what I needed to ask.
36. P: OK.
37. I: I've been kind of struggling with this as I am looking at the curriculum that's out there for middle school students. And point of balance is used a lot. And so I've been struggling with; every expert that I worked with tended to lean more toward the algorithm. So I'm trying to figure out if there is any useful connection.
38. P: That's an interesting question, I think. Because it certainly isn't obvious to me. Is it to the experts that use it?
    I: Yeah, it is obvious that everybody uses it but the connection itself is not – not at all obvious. You know I've actually created a hypothesis in my dissertation (I'm not quite done, I hope I'm going to be done after this break), but I created a hypothesis as to how we might develop it if we wanted it to be developed. But, I'm still struggling with whether it is useful or not . . .

C.2.4 Telephone Interview with B

1. I: You do remember what I'm studying, right?
2. P: You're studying the mean.
3. I: Yeah. So you remember a little bit about the interview then?
4. P: Um – not a lot.
5. I: OK, that's all right. I don't need you to remember anything about it actually. But I have been working on it, trying to analyze the data. And as I have been working, I've come to realize that there were about – well there were a number of different ways that you thought about arithmetic mean. But there were two key ways that you were thinking about arithmetic mean. One way was to use the algorithm for arithmetic mean, or at least the operations that you use within the algorithm. So you did things that were related to adding up all the values and dividing by how many were there.
6. P: OK.
7. I: OK And the second way was that you worked with the arithmetic mean as if it was a mathematical point of balance. So you found a place in the data and you balanced it out.
8. P: OK

9. I: So, does that sound like a fairly good description of how you think you would describe arithmetic mean?
   P: I don't know it is hard to say

10. I: All right. Well, I need to ask you a few questions about those two approaches. OK? And so the question I'm trying to get a little more information about is whether you could explain to me how you understand that the arithmetic mean defined by the standard algorithm and the mean as a mathematical point of balance are the same thing. Do you understand . . . ?
    P: I would probably do it symbolically.

11. I: OK. So it . . .

12. P: I would do an algebraic . . .

13. I: Yeah, where you would make them equal to each other?

14. P: Uh, yeah, that's one way yeah.

15. I: When you do that – what form of the algorithm would you use for the balance?

16. P: What do you mean "form of the algorithm"?

17. I: Yeah, let me rephrase that question. How would you symbolize the half that is the half that you are calling balance? [pause] Am I making any sense at all?

18. P: OK, I'm not sure – how would I symbolize the "half"?

19. I: OK. If you were to take a look at the two equivalent sides, the left hand side and the right hand side. On one side you've got the algorithm . . .

20. P: What do you mean by the algorithm?

21. I: By the algorithm I mean the sum of all the data values and divided by how many points . . .

22. P: OK

23. I: If you were going to symbolically make them equivalent, how would you represent the balance?

24. P: I would start with the balance. Because it would be easier to reduce to the . . .

25. I: OK. So how would you represent that?

26. P: I would make up some letter, use it for mean and take; well let's see, I'd probably take the sum of the deviations equals zero. Would probably be easiest.

27. I: All right. Do you think there would be a way that you could decide, if I were looking at the standard algorithm, of the summation and divided by how many points there are: Is there anything in that that suggests to you that the quantity I'm finding is the point of balance?

28. P: Say again now?

29. I: OK. If I gave you the standard algorithm where you sum the data values and you divided by how many data points there were; is there anything in that algorithm that suggests to you that that should be a mathematical point of balance?

30. P: Well, I think the way I probably think about it is that the mean is a distance, well not a distance but that the distances between the mean and each point balances out. Um, In the actual add them up and divide by the number of numbers you have – as it is I think you'd have to do some transformation. So is there, I mean your wording is "Is there anything in there?" Uh, I think I think about average as this notion of – well I can think of it as leveling off or balancing or . . . [faded off]

31. I: I'll tell you what has really interested me as I've been looking into this. We seem to start learning about arithmetic mean as the result of the algorithm when we first learn about what an arithmetic mean is.

32. P: That's the first thing that people seem to use.

33. I: Yeah. That's what they seem to use first and what has been curious is that the – everyone that I've interviewed has really, quite easily moved to it being a mathematical point of balance. That hasn't been a problem. And the question is how do we develop from this algorithm into understanding it as a point of balance as well?

34. P: Or is it even the right place to start?

35. I: Yes. That's also a question that I'm bringing up. But it's quite interesting. So, let me ask you one more question, if you don't mind. If you had to explain to a 6th grader how the algorithm was related to this idea of balance – in other words, they know that they are supposed to add up the data points and divide by how many data points there are. Is there a way that you think you could describe how that is a point of balance, or is similar to a point of balance?

36. P: To a 6th grader?

37. I: Yes.

38. P: [9 second pause] I think I would probably use a numerical result and look at these and show that the distance from the mean to each of those points balances out.

39. I: Uh huh

40. P: Balances out in that the sum of the deviations equals zero or sum on one side equals the sum on the other.

41. I: OK. Well that was pretty much all I needed to ask . . .

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