EXPLORATION AND INDUCTION VERSUS CONFIRMATION AND DEDUCTION

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Application of the scientific method involves two interacting kinds of reasoning: heuristic exploration and inference to arrive at hypotheses and deductive reasoning to confirm or refute them. Theoretical statistics draws on abstract mathematical reasoning, much of which lies beyond the capability of young students. Nevertheless, they can acquire a sophisticated understanding of statistical concepts by drawing on natural intuitions. There is an analogy in physics with its mathematical basis, which may be understood heuristically. In a similar way, many concepts in probability and statistics have physical analogues. Analogues such as the familiar ball and urn model connect to natural cognitive structures the brain has evolved to enable survival in a complex world. By drawing on these natural cognitive tools, teachers can have greater success both in developing intuitions and in facilitating the formation of abstract concepts.

INTRODUCTION

The theory of probability had its genesis in a 1654 correspondence between Pascal and Fermat (c.f., Gigerenzer, et al., 1990). In the ensuing three and a half centuries, probability and statistics have infused all of science. In the eighteenth century, governments began collecting, tabulating and organizing data about the people as a means to effective governance. These data management activities have grown ever more complex with time. Mendel's nineteenth century theory of genetics brought probability and statistics into the heart of science. By the twentieth century, the study of statistics had permeated every branch of human knowledge. Even physics, the most mathematical of sciences, was not spared: thermodynamics and quantum mechanics adopted probability and statistics at the most fundamental level. Beyond the realm of science, probability and statistics permeate every aspect of everyday life. From weather reports to political polls to newspaper reports on the state of the economy or the latest trends in crime, an educated citizen cannot make it through a day without having to absorb information about probability and statistics.

Recognizing the pervasive nature of probability and statistics and the need for a statistically literate citizenry, educators have begun bringing statistics into the classroom at ever earlier ages. In decades past, a student's first introduction to probability and statistics might occur in secondary or even post-secondary school. Today, there is increasing emphasis on developing statistical literacy in the early grades. Many students engage in activities involving probability and statistics from the earliest grades. Instructional programs have been implemented in many school systems to teach students to collect, organize and analyze data, to interpret results, and to apply the results to decision making (cf., Kader & Perry, 2006).

Nevertheless, the battle is far from won. There remains great variability in the depth and quality of curricula among different school systems (ICMI IASE Study Book, 2010; Kader & Perry, 2006). One of the aims of this paper is to promote the inception of good statistical and probabilistic tools in the earliest grades. Yet, the primary aim is to address the essential differences between teaching traditional mathematics (the mathematics of certainty and deduction, exemplified by geometry and algebra) and teaching statistics and probability (the mathematics of uncertainty).

STATISTICS, STATISTICAL EDUCATION, AND SOCIETY

The past several decades have seen increasing recognition that the context in which probabilistic and statistical learning occur profoundly affects what is learnt and by whom. In traditional mathematics education research, knowledge is conceived as a property of the individual consciousness. The modern concept of stochastic literacy leads us to realize that probabilistic and statistical knowledge, much more so than traditional mathematical knowledge, is grounded in *situations*. This requires us to move beyond analyzing learning as depending solely on a psychological representation of the mind. It is necessary to consider also the setting–its social relationships, its cultural locality, the discursive frameworks available in the locale, the social and

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political environment which frames it–and how that setting functions generatively in the construction of knowledge. In other words, probabilistic and statistical education research has taken a *social turn*.

Stochastic education itself can be understood as a profoundly political activity–in the sense of being intimately tied to environmental and social issues. Sophisticated statistical arguments have been applied to deeply controversial political issues. As an example, consider the spirited controversy over Benford's Law, a statistical rule about the distribution of digits in data generated by natural processes, as an indicator of electoral fraud. Statistically significant violations of Benford's Law have been interpreted by a number of commentators as evidence for electoral fraud in the recent Iranian election (c.f., Roukema, 2009). On the other hand, others have argued (e.g., Carter Center, 2005) that the applicability of Benford's Law to election results has not been established. Evaluating the merit of such arguments requires a basic understanding of probability and statistics, the ability to comprehend statistical arguments, and the ability to evaluate whether the assumptions underlying such arguments apply to a given real-life situation. Statistical arguments are also routinely applied to a large array of questions, including climate change, policies regarding crime, the healthcare debate, the recent financial crisis, and many others. Thus, stochastic education has become a fundamental requirement for an informed citizenry. In addition, conducting stochastic educational research is also a political and moral activity involving issues of values, power, authority and legitimacy.

A FUNDAMENTAL COGNITIVE AND PERCEPTUAL SHIFT

The infusion of probability and statistics into virtually every corner of life has been accompanied by a fundamental shift in how we perceive the world. In the eighteenth and nineteenth centuries, Isaac Newton's grand vision reigned supreme–of a clockwork universe unfolding inexorably according to deterministic laws. The twentieth century brought quantum mechanics, chaos theory, and the realization that even the most sophisticated tools and instruments cannot eradicate indeterminism and variability. From genetics to psychology to engineering, the notion took root that science involved a search for statistical laws manifesting as patterns showing through the noise in empirical data. Today, we look at the world through statistical lenses. We experiment, collect data, perform statistical analysis to search for patterns in the data, construct probabilistic models, and test the models by gathering additional data. This empirical approach has even been adopted for mathematics itself, giving rise to a burgeoning field of experimental mathematics (e.g., Borwein & Devlin, 2008).

Interplay between formal mathematical thinking and heuristic, experimental thinking is best portrayed in the combination of simulation and analytical treatment. Today's instructional settings can successfully combine inference, simulation, modeling and analytical treatment. As an example, consider the famous "Three Door Problem," also known as the "Monty Hall problem," after the well-known game show. Lindenmann (2009) began a series of school interventions in seventh grade by describing the Monty Hall situation: a moderator in a TV show asks someone in the public, which of three doors he should open and explains that behind one of the doors there is a Cadillac while behind each of the other two there is a goat. She represents the three doors on the blackboard and lets the children choose one door. Then, instead of opening the chosen door, the moderator opens another door showing a goat. "Do you stay with your chosen door, or do you decide to change?" Lindenmann reports that some children were silent, while most said the chances stood now 50%-50% and they preferred to stay with the chosen door. The next step was an en-active (in the sense of Bruner, 1960) simulation. Lindenmann formed teams of two who simulated the quiz with yoghurt cups and a coin. She divided the teams into two groups: in one group the choice was always "to stay" with the chosen door while in the other group the choice was always to change. After simulating for a total of 50 times, the groups compared their results. The results gave clear evidence that changing brought more victories.

The next step was a computer simulation, where children could see the results in 10.000 or even 20.000 rounds. Although children "saw" the advantages of changing, they still asked, "*Why*?" This need to ask "why" is one of the most salient aspects of human cognition.

Lindemann explained the problem to her students using an illustration (Figure 1). Each row of the table represents one of the three possible situations, where the grey rectangle represents the car and the white rectangles represent the goat. After the contestant makes a choice, the moderator will open a door, always choosing a door with a goat behind it. We consider the case in which the contestant has chosen Door 1. In, say, 300 such cases, 100 will correspond to Row 1, 100 to Row 2, and 100 to Row 3. For the 100 in Row 1, in 50 cases the moderator will choose Door 2 and in 50 he will choose Door 3. For the 100 in Row 2, the moderator will always choose Door 3; for the 100 in Row 3, the moderator will always choose Door 2. Of these, let us consider the 150 cases in which the moderator chooses Door 2: 50 of them correspond to Row 1, where it is best for the contestant not to change, while 100 of them correspond to Row 2, where it is best for the contestant to change. Therefore, in 2 out of 3 cases, the contestant should change. We can repeat this argument for the case in which the moderator chooses Door 3, and again for the cases in which the contestant chooses Door 2 or Door 3. In each instance, switching doors is better in 2 out of 3 cases. This explanation is not quite a mathematical proof. The real proof would involve a Bayesian treatment. Yet more than 70% of the young students were satisfied with this explanation.

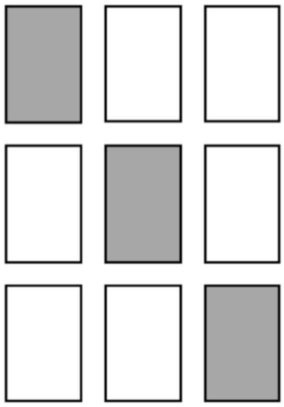

Figure 1.

The Monty Hall problem exemplifies what we mean by experimental mathematics, namely a combination of activities that includes experimentation to support theory building, accompanied by analysis of the theory so constructed. This empirical approach can also be used to build intuitive understanding of traditional mathematical topics, and to teach students about the subtle differences between the statistical and theoretical approaches. For example, students can cut triangles with a scissor from a cardboard and measure the sum of their angles, observing that the measured value is always very close to 180 degrees. Then, the teacher can provide an argument or heuristic to explain why the sum of the angles is always 180 degrees, that is, it adds up to two right angles.

The literature on proof, by Gila Hanna and Niels Jahnke (2004), makes clear to which degree school proofs tend to be heuristic, and similar to physical heuristics. The traditional concept of proof and of a mathematical theory built up from axioms by means of logical inference (Greek mathematics) is beginning to crumble, at least in school education. This is happening not only in Stochastics, but more broadly across the field of Mathematics. The approach now in favor takes inspiration from both the analytical tradition dating back to Euclid’s time, and the pre-Hellenic tradition, as exemplified by the Babylonians or the Egyptians. In the case of probability, as Breiman (1992) wrote: “Probability has a right and a left hand. On the right hand is the rigorous foundational work using the tools of measure theory. The left hand thinks probabilistically: it reduces problems to gambling situations, coin tossing, motions of a physical particle”.

THE RELEVANCE OF STOCHASTICS FOR DECISION MAKING

Teaching stochastics through a combination of experimentation and analysis provides a unique opportunity to apply what is learned to real life decisions. Thus, the age-old question “Why must I learn math?” can be answered: math gives us tools for risk assessment and decision-making. An important and topical example is decision-making about the environment. The movie “An Inconvenient Truth,” presented in schools all over the world, has motivated students to think deeply about environmental issues. For example, the movie was presented as a component of a school intervention in Germany. The movie was presented to ten classes of students of age seventeen, who had already been exposed to the integral and differential calculus and elementary probability and statistics. After the presentation, groups of students were assigned projects in which they developed models for data. One project dealt with CO2 emissions, based on data collected in Mauna Loa (Erickson, 2006). More than 90% of the students participated actively in these projects, even learning additional mathematical tools as necessary for their projects. Students were able to evaluate for themselves the degree to which statements made by Al Gore are to be trusted (Martignon & Sander, 2009; De Haan, Kamp, Lerch, Martignon, Müller-Christ, Nutzinger, 2008). Projects such as this are becoming more frequent in German schools, and form an important part of the Decade for a Sustainable Development (Dekade für die Bildung einer nachhaltigen Entwicklung) launched by the United Nations. They are interdisciplinary, requiring students to draw on knowledge of mathematics, biology, chemistry, physics and other subjects. The paramount importance of tools for dealing with uncertainty is made clear by this type of project.

COGNITIVE INSTRUMENTS TO FACILITATE REASONING WITH UNCERTAINTY

As awareness grows of the importance of acquiring tools for dealing with uncertainty in everyday life and in public affairs, concern also grows about the competence of the citizenry to process uncertainty in a sound and effective manner. An alarmingly large proportion of the public cannot make effective use of probabilistic information. For instance, the German newspaper "Süddeutsche" (Süddeutsche Zeitung Magazin, 31.12.1998) asked 1000 Germans what they think is the meaning of 40%: a quarter, 4 out of 10, or every 40^{th}. Only 54% knew the correct answer, which is "4 out of 10". A burgeoning literature has documented disparities between the results of unaided judgment and the prescriptions of the probability calculus (Kahneman, Slovic and Tversky, 1982). As Gould (1992) summarized: "Tversky and Kahneman argue, correctly, I think, that our minds are not built (for whatever reason) to work by the rules of probability." On the other hand, many negative results on human reasoning under uncertainty can be explained by discrepancies between the environments and tasks faced by our ancient forebears, and those given to subjects in psychology experiments (c.f., Gigerenzer, et al., 1999). One might ask, then, whether uncertain reasoning skills could be improved by exploiting, rather than fighting against, the reasoning strategies humans find natural. Beginning in the early grades, teachers could match pedagogical strategies to children's natural cognitive processes, building skill at solving concrete problems with clear physical analogues. Later, when more abstract thinking skills are introduced, students can be taught heuristics for translating these abstract problems into representations that are by that time well internalized.

Research results on the cognitive mechanisms underlying probabilistic reasoning suggest the potential effectiveness of such an approach. Consider the important *Bayesian reasoning* task, in which evidence about an uncertain proposition is used to revise one's assessment of the likelihood of a related proposition. Zhu and Gigerenzer (2006) devised an experiment to test children's Bayesian reasoning skills. They gave children a story about a small village in which liars tend to have red noses. The story contained information about the prevalence of liars in the village, the chance that a liar has a red nose, and the chance that a non-liar has a red nose. After hearing the story, children were asked to assess the chance that someone with a red nose is a liar. When the information was given as natural frequencies, fourth grade children performed better than they did when the information was given as probabilities (see Figure 2 for an example of a Bayesian task presented in the natural frequency setting). Zhu and Gigerenzer theorized that human cognitive processes may be adapted to the natural frequency representation, which can be understood as mental simulation of counting. Atmaca and Martignon (2004) conjectured that the natural frequency and probability versions of the Bayesian task make use of different neural circuits. In experiments to evaluate this hypothesis, subjects mentally solved the two versions of the task. Atmaca and Martignon found that subjects given the natural frequency format produced results in significantly less time and with significantly more correct answers than those given the probability format. Figure 3 shows results of another experiment, in which fourteen-year-old students were required to solve Bayesian tasks with three or four branches (examples are shown in Figure 2), and had to produce results in a very short time interval.

***Bayesian task with 3 branches**: 10 out of 1000 children have the German measles. Out of the 10 children who have the German measles, all 10 have a red rash. Of the 990 children without German measles, 9 also have a red rash. How many of the children with a red rash have the German measles?*

***Bayesian task with 4 branches**: 10 out of 1000 car drivers meet with an accident at night. Out of the 10 car drivers who meet with an accident at night, 8 are drunken. Out of the 990 car drivers who do not meet with an accident at night, 40 also are drunk. How many of the car drivers who are drunken actual meet with an accident at night?*

Figure 2.

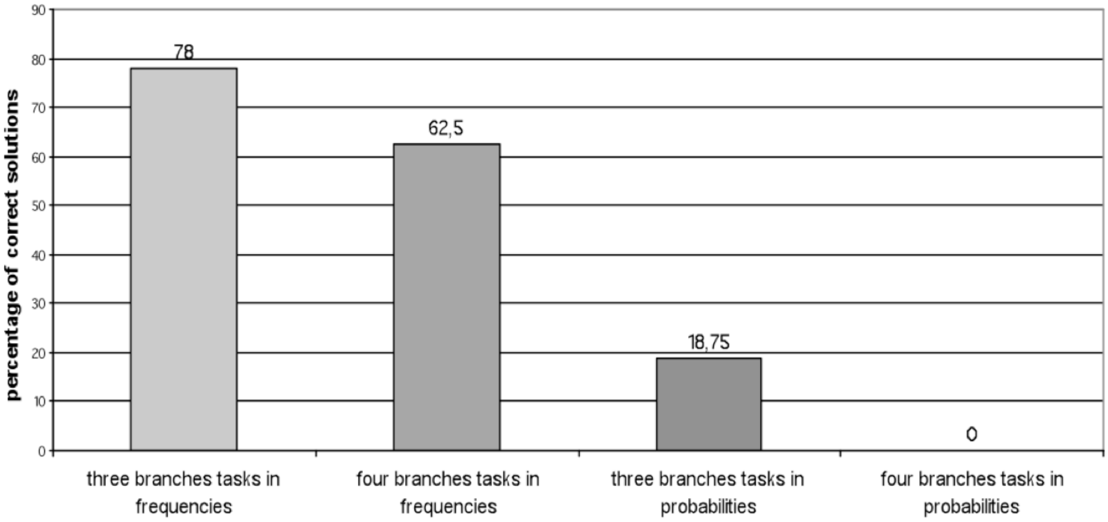

Figure 3. Accuracy of Subjects as Function of Problem Complexity and Presentation Format

IMPLICATIONS FOR PEDAGOGY

Traditional mathematics education differs fundamentally from stochastic education in that the primary goal of the former is to transmit knowledge of mathematical entities *per se*. While these entities may have deep connections with real world problems, the primary context of traditional mathematical learning is "mathematical" and thus "intrinsic". The primary context of stochastic education is dealing with data and thus extrinsic in nature.

Populations and samples become central themes by the third and fourth grade, generating the need for appropriate pedagogical tools. An effective technique is analogue modelling of populations and samples using manipulative such as Tinker Cubes and Tinker Towers. These are colored cubes that can be put together to represent individuals and populations. Tinker Cubes can be used with materials such as the book "If the world were a village" (Smith & Armstrong 2003). This book provides a thoughtful quantitative description of different populations and subpopulations of the world. Designed for young children, it can be used in the third and fourth grades. The book asks the reader to imagine the world as a village of 100 people. Chapters of the book are devoted to sub-populations of this world-village, divided according to features such as nationality, language and religion. For instance, the first chapter treats the continents of the world: 61 of the 100 people are Asians; 12 are Europeans; 5 are North Americans; 8 are South Americans; 13 are Africans, etc. Teachers can create experiments in which children construct towers of Tinker Cubes for encoding specific features (e.g., snap together a yellow cube for European and a brown for Christian to obtain a Christian from Europe). Students are then able to answer questions of the type "How many Christians in the village are Europeans?" and "How many Europeans are Christians?" Modelling the world village by means of Tinker Cubes is a first step to learning to use statistical tools for an understanding of the political and social environment surrounding us.

Another major issue that demands statistical literacy is medical decision-making. Children can begin early to develop the basic quantitative literacy to enable them to evaluate, for example, the diagnostic value of a positive medical test. These skills can be taught by modelling populations according to features and counting to answer questions like "Would you bet that a child with long hair is a girl?"

CONCLUSION

Statistical literacy has become a necessary survival skill for twenty-first century society. Statistical reasoning involves the interplay between empirical, heuristic reasoning and deductive, analytical thinking. By understanding the natural modes of cognition with which evolution has equipped the human mind, and devising pedagogical strategies that build on these modes, teachers can pave the way toward the development of more abstract, analytical thinking in later grades.

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