COIN-SEQUENCES AND COIN-COMBINATIONS TAUGHT AS COMPANION TASKS

Anthony Bill & Peter Gayton
Faculty of Education, University of Tasmania, Australia
Department of Education, Tasmania, Australia
afbill@utas.edu.au

Using both physical and Fathom™ virtual coins Year 10 students explored the distinction between coin-sequences and coin-combinations (where the order is not important) presented as companion tasks within an integrated coin system. This approach sought to highlight two key concepts within the one teaching unit: coin-sequences, to examine independent events, and coin-combinations, to examine the binomial distribution. The misconception that a particular coin-sequence (e.g. HHHHH) is less likely to occur may arise when a coin-sequence task is re-interpreted incorrectly as a coin-combination (e.g. 5 Heads). Pedagogy was based on principles that emphasised profound understanding through the use of multiple approaches to problem solving.

This study examined Year 10 students' responses to an exploration of the binomial using the three methods of the binomial formula, Pascal's Triangle, and a *Fathom*™ simulation (Key Curriculum Press, 2005), and whether a multiple-method approach assisted student learning. Selected aspects of this study, such as the introductory physical and virtual "coin" activities presented here, gave students the opportunity to explore the distinction between an ordered coin-sequence and a unordered coin-combination, develop confidence in the Fathom simulation as a legitimate mathematics tool, and provide a foundation for a formal study of the binomial.

THEORETICAL BACKGROUND

Probability is a topic where students are likely to bring their own beliefs and misconceptions to the classroom–misconceptions that are notoriously resistant to change and may explain the existence of learning difficulties (Batanero & Sanchez, 2005). These misconceptions may include the belief that a coin-sequence of HHHHH is less likely than a sequence of HHTHT.

Students' development of understanding of a multiple coin toss or related tasks have been a topic of research for many years (e.g., Kahneman & Tversky, 1972). More recent studies (e.g., Konold & Kazak, 2008) examined combinatorial tasks where middle-school students constructed the distribution sample space using computer simulation. With the notable exception of Abrahamson (2009) this research considers either sequence or combinatorial concepts–but not both–rather than students' development of a holistic understanding of a coin system.

Pedagogy for the research was informed by statistics education research best-practice including experiential tasks, whole class discussion, and use of appropriate technology, and by principles identified by Ma (1999) that she believed were the essential distinction between Chinese and American elementary mathematics teaching: (a) basic ideas, explicitly taught and reinforced; (b) multiple perspectives to solve problems; (c) a connectedness, to avoid fragmenting students' learning; and (d) longitudinal coherence, where critical concepts taught earlier are linked to what students will learn in subsequent years.

METHOD

Sample

The convenience sample was a Year 10 class in a government all boys school in Tasmania, Australia. The group of eighteen students was an advanced mathematics class, but the students were self-selected and presented with a range of abilities. The classroom component of the study was taught by the authors as a naturalistic teaching program over a ten day period in 2008 within a broader research program into the use of Fathom in schools.

Procedure, tasks, activities, and data collection

Paralleling and extending the work of Abrahamson (2009) this study sought to address the perceived short-coming of existing research by exploring the distinction between a specific ordered coin-sequence and a coin-combination, presenting the activities as companion tasks within the one teaching unit. This is not an innovation: the two tasks were part of the 1960s curriculum. Such an

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approach served a three-fold function: (a) provides a natural point of departure for students, (b) investigates the coin system thoroughly, and (c) introduces a formal study of the binomial theorem. The teaching unit examined the binomial theorem using the three approaches of the binomial formula, Pascal’s Triangle, and Fathom, but only data from two items from the pre- and post-tests, activities and discussion from the first lesson, and items from a post-study attitude questionnaire are presented here. Consistent with Ma (1999) simple, but fundamental, concepts were investigated intensively from multiple perspectives with a context carefully chosen so as not to obscure the underlying mathematics. The term physical coin (as opposed to real) was used to promote the idea of equivalence with the virtual coin, and the term coin-sequence was used as a natural language alternative to permutation.

Pre-and post-test Task 1: Coin-sequence

The principal objective was to determine students’ preconceptions of independent events.

Which of the following sequences is more likely to result from flipping a fair coin 5 times: (a) HHHTT; (b) HTTHT; (c) THTTT; (d) HTHTH; (e) all four sequences are equally likely? Explain your answer.

The correct response is (e), and an example of the preferred explanation might be: “the outcome of the toss of a fair coin is equally either heads or tails”

Pre-and post-test Task 2: Coin-combination

The coin-combination task examined the most likely outcome of flipping a coin five times, but with only the total number of Heads or Tails recorded.

A fair coin is flipped five times and only the total number of Heads and Tails is recorded, not the sequence in which it occurred. Which result is more likely: (a) 3H&2T; (b) 5H; (c) H&4T; (d) all three of (a),(b) and (c) are equally likely; (e) it is impossible to give an answer. Explain answer.

The correct response is (a), and an example of the preferred explanation might be “the 3H&2T combination is more likely because it can occur in many different sequences”.

Activity 1–Manual coin toss where students worked in pairs to toss a coin twice, and the process repeated twenty times. The frequency that each sequence occurred was calculated, re-expressed as a coin-combination, and both were recorded on a poster to promote class discussion (Figure 1).

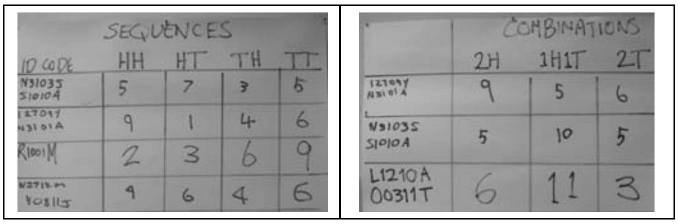

Figure 1. Physical coin tossed twice, and repeated 20 times – extract only

Activity 2–Guided class discussion, using informal language, sought to highlight key features of the data: each sequence is equally likely (independent events), but in practice this result is unlikely; data from one student pair may be inconclusive or even contradict theory, and that data from other students is needed; “neat” (HH & TT) sequences are as likely as “mixed” (HT & TH) sequences; mixed combinations are created by summing the frequency of constituent sequences; the likelihood of a mixed combination is higher because it is can arise from many different sequences, but the individual sequences remain equally likely; the “ends” of the sequence and combination data have the same frequencies; in everyday language the distinction between sequence and combination is rarely made, and this might explain why people believe the sequence HH is less likely because they “see” it as the combination 2H, which is indeed less likely than other mixed combinations.

Activity 3–A virtual coin toss using a Fathom simulation introduced students to Fathom–software provided by the industry partner, but suitable for the student group. The virtual simulation was purposefully designed to mimic the physical simulation and the data presentation format used previously, and pre-constructed so that a direct connection could be made between the physical and virtual simulations within the one lesson. Rather than constructing their own simulation the students were assigned three essential tasks: first, methodically proving that the simulation modules (collection, sample, and measures) functioned correctly; second, confirming that the various data representations were internally consistent; and third, comparing the results with their own intuitions of a physical coin–the justification for this approach with this young male cohort was test-driving a car before purchase (Figure 2.)

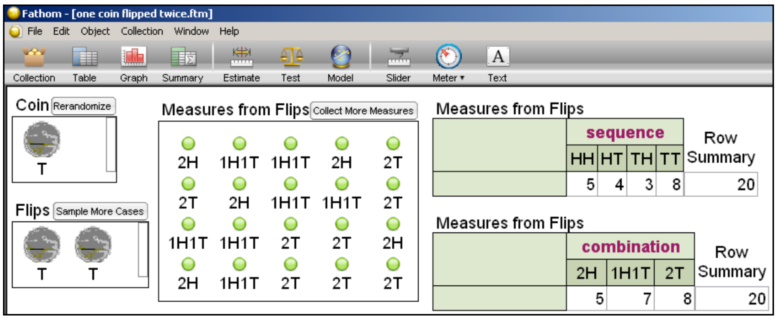

Figure 2. Fathom™ virtual coin tossed twice, and repeated 20 times

RESULTS

Task 1: Coin-sequence

Students' performance on the task in the pre-test (94%) was very high, with students providing an explanation that each outcome, heads or tails, was equally likely. This was a higher level of correct responses than that found by Konold, Pollatsek, Well, Lohmeier and Lipson (1993) in a study of first year pre-service teachers (86% correct), and it is consistent with their conclusion that by early teens the majority of people believe in the independence of trials of coin flipping. Students' performance in the post-test was unchanged.

Task 2: Coin-combination

Students' performance on this task on the pre-test (39%) improved on the post-test (67%), but remained considerably lower than the companion coin-sequence task throughout the study. The persistence of student beliefs noted by Batanero and Sanchez (2005) was clearly evident: of the students who gave an incorrect answer on the pre- and post-test (33%), all but one gave the same incorrect answer that (d) "all combinations are equally likely" on both tests; and all the students who gave a correct response and explanation on the pre-test gave the same correct response on the post-test. Students demonstrated development of understanding with 22% providing an incorrect response on the pre-test, but a correct response on the post-test; and a further 28% who gave a correct pre-test response extended their response to include a gratuitous formal calculation on the post-test. Intriguingly, students who gave an incorrect post-test response to this item demonstrated competence in other, more complex items, on the same post-test.

Activity 1 & 2: Manual coin toss and guided whole class discussion

Students had little difficulty with the manual coin toss and calculations, and most participated enthusiastically in the discussion. The strength of students' responses during the discussion suggested the distinction between coin-sequences and coin-combinations was, at that stage, clear.

Activity 3: Virtual coin toss using a Fathom simulation

Students responded to their first exposure to the simulation as anticipated: healthy scepticism (this was actively encouraged as part of cultivating acceptance). To several students the simulation was a video game, with repeated–almost mindless–sampling to obtain what they described as a "perfect result" where each sequence occurred with the same frequency. Ownership, developing confidence, and making the transition to the virtual world took several classroom hours. In subsequent activities the students constructed their own simulations, and by the conclusion of the ten day study Fathom was, as shown by a post-study student questionnaire, accepted as a legitimate mathematics tool.

DISCUSSION

Probability education research recommends activities designed to confront misconceptions: Task 1 and 2 are such activities. Students' understanding of Task 1 the five coin-sequence was robust, but students' performance on Task 2 coin-combination remained below that on Task 1 coin-sequence throughout the study. A coin-combination task is arguably more intuitive than a coin-sequence task, and it is intriguing that students' performance was lower on what might be a simpler task. Their performance could reflect prior experiences at school that emphasised independent events rather than coin-combinations, or casual conversation, where the distinction between coin sequences and combinations is made rarely. A possible explanation for students' incorrect responses may lie with a re-interpretation of a coin-sequence to a coin-combination (or vice-versa). This re-interpretation occurs because it is easier to describe a coin-sequence e.g. HHTHT as the combination 3H&2T. The students then responded correctly to the re-interpreted task–the underlying error lies with the re-interpretation of the task, not with the actual response. This re-interpretation might have been exacerbated by the order in which the two tasks were given. Students' incorrect response to the post-test coin-combination item, but good response to complex procedural tasks, demonstrated either their incomplete understanding, or that their attention had shifted to procedures. Simple revision of the Fathom simulation may have addressed this difficulty.

CONCLUSION AND IMPLICATIONS FOR TEACHING AND RESEARCH

Students' understanding of the coin-combination task was persistently less well-developed than their understanding of the companion task of coin-sequences, and many students continued to confound the two. Misconceptions may arise when a coin-sequence is re-interpreted as a coin-combination, or vice-versa. Schools may presently fail to make this distinction clear. Coin-sequences and a coin-combinations should be considered companion tasks in a holistic examination of a coin system: coin-sequence tasks allow exploration of independent events and the more intuitive coin-combination task provides a basis for a study of the binomial. The use of multiple methods to examine conceptually simple tasks in-depth was acknowledged by the students and the colleague teacher, and shown by the post-test responses to the questionnaire, as an effective teaching strategy that may develop students' profound understanding of concepts. Building from a familiar physical activity and purposefully developing confidence in the virtual simulation may be key first steps in facilitating a transition from physical to virtual simulation. Fathom, a sophisticated tool, can be used effectively in schools to examine fundamental concepts.

Fathom seemed valued more by the less able students than by the strongly procedurally competent. This response, and the high level (94%) of correct answers to Task 1 pre-test, suggests further research on these topics using the same approach, but with less able students, is warranted.

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REFERENCES

Abrahamson, D. (2009). Orchestrating semiotic leaps from tacit to cultural quantitative reasoning–The case of anticipating experimental outcomes of a quasi-binomial random generator. *Cognition and Instruction*, *27*(3), 175–224.

Batanero, C., & Sanchez, E. (2005). What is the nature of high school students' conception and misconceptions about probability? In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 241–266). New York, NY: Springer.

Kahneman, D., & Tversky, A. (1972). Subjective probability: A judgment of representativeness. *Cognitive Psychology*, *3*, 430–454.

Key Curriculum Press. (2005). Fathom Dynamic Data Software. (Version 2.1) [Computer software]. Emeryville, CA: Author.

Konold, C., & Kazak, S. (2008). Reconnecting data and chance. *Technology Innovations in Statistics Education*, *2*(1), 1–37. Online: repositories.cdlib.org/uclastat/cts/tise/vol2/iss1/art1.

Konold, C., Pollatsek, A., Well, A., Lohmeier, J., & Lipson, A. (1993). Inconsistencies in students' reasoning about probability. *Journal for Research in Mathematics Education*, *24*(5), 392–414.

M., Liping. (1999). *Knowing and teaching elementary mathematics*. Mahwah, NJ: Erlbaum.