DUALITY OF PROBABILITY AND STATISTICS TEACHING IN FRENCH EDUCATION

Pablo Carranza and Alain Kuzniak
Université Paris Diderot, France
pfcarranza@gmail.com, kuzniak@tele2.fr

In epistemological studies, two main interpretations of probabilities are the frequentist and the Bayesian. In this paper we first show that both interpretations are present in French Secondary schools, albeit the official curriculum only supports the frequentist approach. We then suggest a possible teaching situation to introduce teachers and students to the subjective-objective duality with some statistics training situations via the use of problem solving. We also give some characteristics and conditions useful to build situations fitted to this goal.

INTRODUCTION

In this paper, we are interested in the teaching of statistics viewed as a set of methods leading to rational decision-making in a context of uncertainty. More exactly, we study how two main statistical interpretations of probabilities, frequentist and Bayesian, are handled and their impact on the teaching of statistical inference in French schools. We will first briefly describe each of these interpretations of probability as used in our work. Then, we will analyze the approach currently adopted in the academic curriculum and in textbooks in France. Then, we present an example of a possible teaching situation involving both interpretations that we have devised. Some of the features involved in such a situation will be listed with the aim of supporting the subjective grasp of the probability.

DUALITY OF STATISTICAL INTERPRETATION

For our study and from a didactic viewpoint we will consider two distinct *signifiés* of probability*:* the frequentist and Bayesian interpretations. The notion of *signifié* we use comes from educational studies in semiotics and mathematics and could be roughly translated by "meaning" or "connotation." Presmeg (2006) used the word signified, but we will keep the French word in this paper to avoid confusion. Since the end of the 17th century, the "taming of chance" (Hacking 1990) has been following two paths: Frequentist and Bayesian. In the frequentist interpretation, the probability of an event is the theoretical limit of frequencies. The relative frequency of a given outcome gradually stabilizes when the number of identical events or experiments producing this outcome increases. This first notion is justified by the weak law of large numbers upon which the frequentist interpretation is based. In the Bayesian approach, a probability is a measure of belief in a certain state of affairs given a certain state of knowledge. Probability is not an intrinsic characteristic of the object but a measure of certainty (a degree of belief), given by a subject to a proposition. The probability initially attributed to an event can differ from one observer to another, and this subjective view poses the question pointed out by Batanero, Henry and Parzysz (2005, p 34): "What is the scientific stature of the results which depend on judgments that vary with the observer?". As they noticed, the solution to this question depends on the status given to the mathematical model of probability theory. To evaluate the probability of an event, the notion of conditional probability is used and this probability evolves in a convergent way as new information is handled using the Bayes' theorem, which is here the key theorem.

In practice, most of the time there is a failure to differentiate between the two interpretations of probability, which are spontaneously combined in statistical situations where they are applied simultaneously. We therefore posit that the *signifié* of probabilities has a dual character, and we shall speak of "intricacy of the *signifié*," to keep the idea of confusion and mingle included in the duality. For some authors who agree with this complexity, the main reason for this intricacy is the exclusive reliance upon the frequentist principle and long-term expectation as the unique basis for the degree of certainty (Gärdenfors & Sahlin, 1988; Hacking & Dufour, 2004; Shafer, 1992), while ignoring the intuitive/subjective information when assigning a probability on the basis of prior knowledge.

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Starting in 2000, more emphasis has been placed on the learning of statistics in French secondary schools, and the mathematics teachers have to teach statistics even though they are minimally trained in this domain. French mathematics teachers have a high level in mathematics but scant knowledge of statistics, which is optional for graduate mathematics students. So, they regard statistics as a direct application of probability, and in their teaching they generally try to transform statistics problems into formal probability problems that involve technical calculations but no interpretation. Furthermore, the official Education Ministry documents favor exclusively the frequentist statistical interpretation and propose exercises where the frequentist approach appears to be the only one needed to solve problems.

In this context, pre-service and in-service teacher training is crucial for the purpose of introducing a new approach to the teaching of statistics and avoiding the kind of didactical obstacle described by Brousseau (1997, p. 86) as "obstacles of didactical origin are those which seem to depend only on a choice or a project within an educational system." Moreover, the official documents written by the Education Ministry only promote the statistical interpretation and suggest exercises where the frequentist approach would apparently be the only one needed to solve the problems. Our basic hypothesis is that the duality of *signifié* of probability is intrinsic to the notion of probability and that these two interpretations cannot be separated. This means our assumption is questioned by the official approach to teaching statistics today that intentionally separates these two aspects. If this teaching approach were successful it would be strong evidence that our hypothesis is wrong.

In order to check this hypothesis we studied the French secondary school textbooks at Grade 11 to see whether textbooks really separate the two meanings and reduce statistics to only the frequentist approach and whether their reduction is successful. We used implicative statistics analysis (Gras et al., 1996) to study the potential performance in solving the exercises proposed in textbooks. A set of 21 variables was considered to identify the underlying *signifié* in the solution of exercises. The following selection of variables illustrates the notion under consideration:

- Are the model hypotheses explicitly given in the exercises or do they have to be derived from the data given in small or large samples?
- Is the calculation of probabilities explicitly requested or not?
- Is there a frequentist context unequivocally described or not?
- Is there an interpretation requested or not?
- Is the event unique or reproducible? Is it specific or generic?
- Does the exercise clearly address a single, random event? Is it taken from the real world?

The remaining variables were more descriptive and gave technical information about the exercises (e.g., length of the wording, number of questions).

The presence of the "intricacy of the *signifié*" was confirmed by our analysis even when the stated intent in the curriculum is to reduce the complexity of the *signifiés* to the sole frequentist approach and in spite of some differences between the scientific and economic sections in secondary schools. In the sciences section, the effort centers on computer techniques based on one axiomatic model to the detriment of the *signifié* of the probability calculated. The approach is essentially computational (*opératoire*) (Duval, 1993). Almost all exercises are given in a formal mathematical context that does not leave room for statistical interpretation. In the economic section, the working space is less strictly defined and could appear as "fuzzy", and there is greater expectation for interpretations. For students outside of scientific studies sections, the real world takes a more important place in these exercises, and the context is less formal than in the former section.

In both sections, some exercises are Bayesian in nature, yet an interpretation is not requested, and the Bayesian notion remains hidden from the students. Frequentist exercises, on the other hand, are rare and they are very long, which underscores the difficulty of building short situations within a frequentist context. As it takes too much time for their resolution in the classroom, these exercises are placed at the end of chapters as optional activities. In contrast to the Bayesian exercises, an interpretation is explicitly requested, which could allow for a reflection on the *signifié.*

This short study supports the validity of our hypothesis on the "intricacy of the *signifié*" in the approach to the teaching of probability and statistics in France. The intricate nature of the *signifié* appears in some exercises proposed by textbooks even though the official curriculum denies it. Thus, the concept of probability is truncated: the frequentist definition is the only approach taught, while students are confronted with frequentist and Bayesian problem situations. To avoid this difficulty, teachers are doing technical side-stepping; they ask the students to do the computations while questions regarding interpretations are left aside and the *signifié* is ignored. According to the position of probability within a statistics curriculum described by Borovcnik and Peard (1996), we can say that mathematics teachers are viewing statistics through the axiomatic viewpoint (p. 257) whereas the official curriculum promote the naive frequentist interpretation of probability insisting on simulation (p. 256).

TOWARDS A TRAINING OF TEACHERS INVOLVING THE DUALITY OF INTERPRETATION

So, even if the intricacy of *signifiés* is present in the classroom, it is officially ignored (hidden) by the institution. Moreover, beyond the presence of the intricacy of *signifies,* our study shows that the French approach to statistics is characterized by a lack of any kind of statistical interpretation. This suggests that we should devise problems specifically designed to facilitate the emergence of the two interpretations and highlight some of the requirements in developing such problems.

In order to fulfil this need we started a research project where we investigated a number of variables needed to determine a specific *signifié* and developed the corresponding teacher training situations. We briefly report below about one problem based on the well-known bottle situation worked out by Brousseau and colleagues (2001) and used by other authors (Briand 2005) in a frequentist context. We have modified this situation to introduce a Bayesian context.

The problem has two parts: The first section takes place in a Bayesian context and the second, in a frequentist context. In the first phase, an opaque and closed bottle with a transparent top and containing four balls, each of which is either orange (O) or black (B), is presented to students. By tipping the bottle upside down, it possible to see one ball in the top. We ask the students to find a method for estimating the color composition of the bottle content without opening it. They are asked to express their belief in each of five possible color compositions using 20 coins, according to the following rules:

a. The stronger you believe in a color composition, the more coins you bet on it;
b. If you believe more in one color composition than in a different one, you must justify your belief.

For example, if you bet no coin to the composition BBOO, then you are sure that the bottle content is not of the type BBOO. Conversely, if you put all the coins on the composition BOOO, you are sure that the bottle content is of the type BOOO. Participants were asked to write the initial probabilities (degrees of belief) and the change in these probabilities after they overturn the bottle and see the ball and its color in the transparent top. They can use spreadsheets for calculation. This part was based on Bayes' Formula implemented on the computer and focused on decision making.

During the second phase of the exercise, approaching the problem from a frequentist perspective, participants could replicate the event directly with transparent bottles or through computer simulations. For this, they introduced the composition of the bottle they wanted to test and had to overturn many times the bottle to draw the curve of relative frequencies.

In designing and experimenting with this type of situation, we established a set of conditions and variables required in order for the teachers to define the context of the problem and facilitate that the treatment of the problem that supports a particular *signifié* of probability. Below we describe four of these variables for the particular example of the bottle problem.

The indeterminist choice

In designing the situations we retain the philosophical hypotheses connected to the fundamental question of indeterminism and chance. To change students' approach to statistics and make them grasp the paradigm of indeterminism, we have to choose problems based on real situations such as using a bottle as a randomization device rather than using immediately a simulation based on a probability law.

Decision-making

We compel students to make decisions by requesting that they bet on a specific content configuration for the bottle. They have to use probabilities to make their decision (Hacking 1975). Moreover, students have to explain their decision and justify it verbally. The transition to the discursive semiotic register (Duval 1993), here the French language, creates the conditions for developing the signifiés.

On the other hand, a high degree of precision is needed in solving the problem as well as a computation of probabilities. After working in the discursive register, students have to use the symbolic register, and at the end of the session they interpret their result in natural language.

The nature of objects

By definition, the frequentist interpretation implies that the probability *P (A)* is only defined for events that can be replicated many times and that this value expresses their long-term tendencies. In such cases, reference to a long-term expectation must be made explicit, and it is not possible, for instance, to inquire about the color of one card hidden under the teacher's hand, a case depending on a Bayesian approach. In this latter example, the domain of application of *P (A)* is less restrictive, and unique occurrence cases are allowed (for instance see Hacking & Dufour, 2004, p. 148).

Type of logical reasoning

We finish with a difficulty that French mathematics teachers encounter when faced with statistical problems and their use of deductive reasoning. Bayesian reasoning is based on abduction in the sense of Peirce (Burks 1946): A property is asserted on the basis of a little information (sometimes only one datum) deemed sufficient to ground belief. Teachers find it easier to accept the frequentist interpretation, which is commonly perceived as a deductive reasoning (e.g., the law of large numbers), even though the expectation of stabilization of the frequency in the system considered could result from inductive reasoning.

CONCLUSION

The problem described in our experiment can be solved with the mathematical knowledge acquired by the end of the French secondary education cycle (17-18 years old). However, the teaching of statistics is lacking subjective probability based situations, because working with these problems proved to be challenging. We summarize some of the reasons:

a. Teachers and students are asked to give *P(A)* a meaning in the "real world" in contrast to the usual practice of context free and predefined situation in the teaching of statistics in France.
b. To manage decision making situations, teachers must be familiar with the intricacy of the probability *signifié*, but this is not sufficient; they also need support from educational institutions that should provide these teachers with teaching situations with an emphasis on decision-making.
c. Students use poor language to deal with the notion of probability (e.g., they misuse words such as chance, rare, probable).
d. Students find it very difficult to match the notion of indeterminacy with decision-making. They were mystified by the uncertainty that comes together with their choices (they only knew the composition of the bottle content in terms of probability). In other words, this new kind of reasoning did not suit students, and some of them did not take it seriously and asked that the bottle be opened. This reasoning may also challenge too many teachers, and sometimes they may not resist the request that the bottle be opened.

In sum, one of the main challenges to teaching different statistical interpretations of probability comes from mathematics teachers with little training in statistics and who resist a reasoning that is quite different from the traditional mathematics way of thinking. We suggest that working with these teachers on discussing the specific characteristics of the context is needed if we want them to grasp one or another statistical interpretation of probability and to get over the problem of teaching subjective probability.

But it is not enough to change the didactical contract, and we have seen that pre-service teachers need to experiment with such an approach. Therefore, we suggest developing training situations for teachers, such as the bottle experiment described in this paper, where, at the same time, student teachers can experiment and reflect about what they have to teach.

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