Graphical Perception as a Basis for Drawing Graphs

Ross Ihaka
University of Auckland

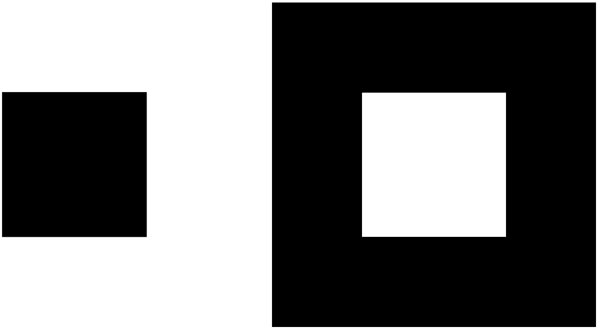
Outline

- Scope of the talk
- Colour choice for area fills
- Drawing smooth curves
- Extension to piecewise smooth curves

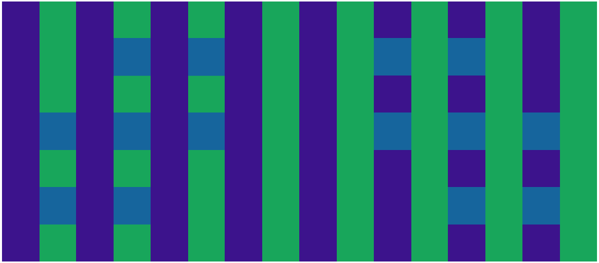
Colour

- Choosing colours that work together is **hard!**
- Relying on “judgement” or “taste” is very unreliable for most of us.
- Choosing colours for data graphics is even harder, because badly chosen colours may not just look horrible, they can also be misleading.

The Irradiation Illusion



Simultaneous Contrast



Colour Theory 101

- The normal human eye has three colour receptors with peak sensitivities in the red, green, and blue regions of the spectrum.
- A sensation of colour can be produced stimulating the red, green and blue receptors with pure (single wavelength) colours at the peak sensitivity of each kind of receptor.
- This permits an external description of colour by the amount of each primary used to produce the colours.
  - RGB: the amounts of red, green, and blue light used to produce the colour.
  - HSV: a mapping of RGB into hue, saturation and value that correlates better with our perception of colour in terms of hue (colour of the rainbow), purity (red vs pink) and brightness (light vs dark) of colours.

Colour Theory 101

- A better understanding of colour can be gained by understanding how the vision system processes colour.
- Internal circuitry converts the signals from the receptors into light-dark, yellow-blue and red-green signals.
- Transformation of the yellow-blue and red-green signals to polar coordinates converts them to our perceptual notions of colour-hue and colour purity.
- The mapping of colour stimulus to perceived colours is highly non-linear.
- It is useful to try to mimic this non-linearity to match the way that we “see” colours.

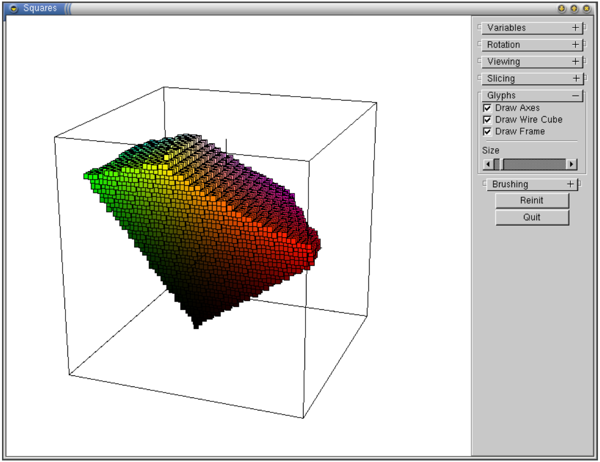
Perception of Colour

- The use of RGB and HSV spaces does not yield a faithful representation of the way in which we perceive colour.
- It is useful to apply a non-linear transformation to the three-dimensional colour space to make it more perceptually uniform.
  - Euclidean distances in the transformed space provide a measure of perceptual distances.
  - Pairs of colours the same distance apart in the space will appear "equally similar."
- A number of spaces have been defined which provide "perceptually uniform" representations of colour that can be transformed in RGB or other colour specifications.

Uniform Colour Spaces

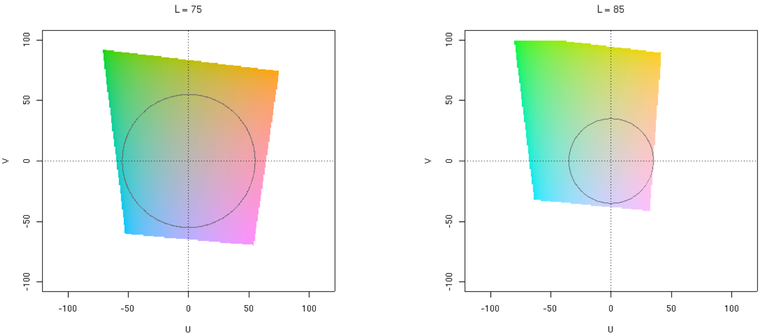
- CIELAB is used in the printing industry (subtractive colour applications)
- CIELUV is used in lighting applications (additive colour)
- CIEUVW is an attempt to be even more perceptually uniform.
- Use of these spaces makes it possible to specify colour in a way that “matches” human perception.

LUV Space



Squares
Variables
Rotation
Viewing
Slicing
Glyphs
☑ Draw Axes
☑ Draw Wire Cube
☑ Draw Frame
Size
Brushing
Reinit
Quit

Horizontal Slices of LUV



Colours on each of the circles differ only in hue.

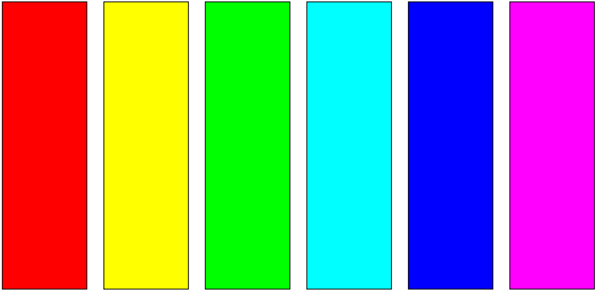
Choosing Fill Colours

- Choose a set of colours that are equally bright and equally colourful.
- This ensures that we view the colours without bias.
  - Equal brightness eliminates illumination illusion effects.
  - Equal colourfulness means the eye is not preferentially drawn to particular areas.
- There is also a useful side-effect of choosing colours in this way . . .

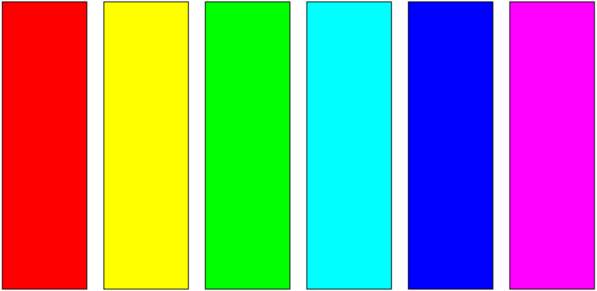
Colour Harmony

- Choosing colours that “work together harmoniously” is hard.
- A few people have an innate ability to do this, but most of us don’t.
- In the course of teaching painting at art school, Albert Munsell quantified a set of rules for producing harmonious colours.
- The essence of these rules is to choose colours equally spaced along smooth curves in a perceptually uniform colour space.
- The proposed method of choosing colours fits this principle.

Equally-Spaced HSV Colours



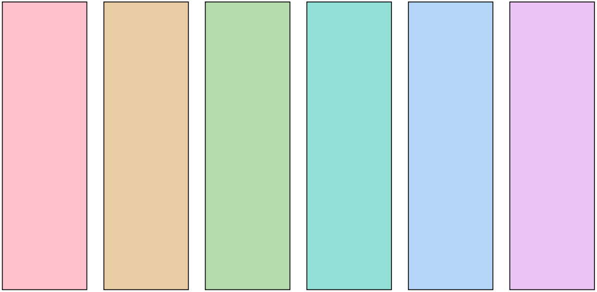
Equally-Spaced HSV Colours



"The circus wheel and poster, although they yell successfully for momentary attention, soon become so painful to the vision that we turn from them."
— Albert Munsell

Equally-Spaced Harmonious Colours

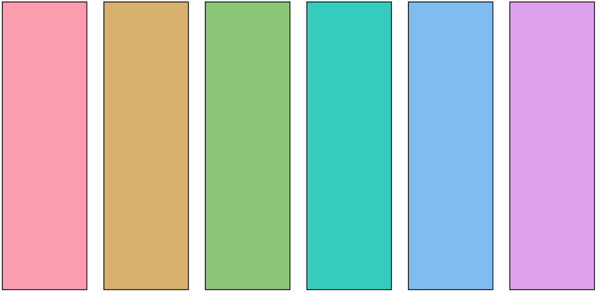
The following colours were chosen using the R `hcl` function.



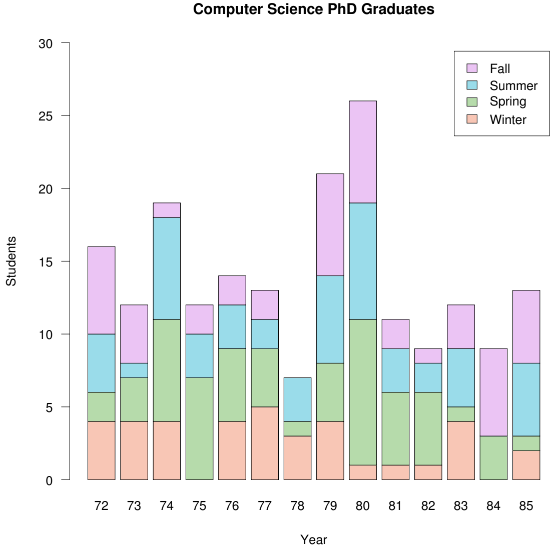
A Six Colour Palette (L = 85, C = 35)

Equally-Spaced Harmonious Colours

Darker, more colourful colours chosen using the R `hcl` function.

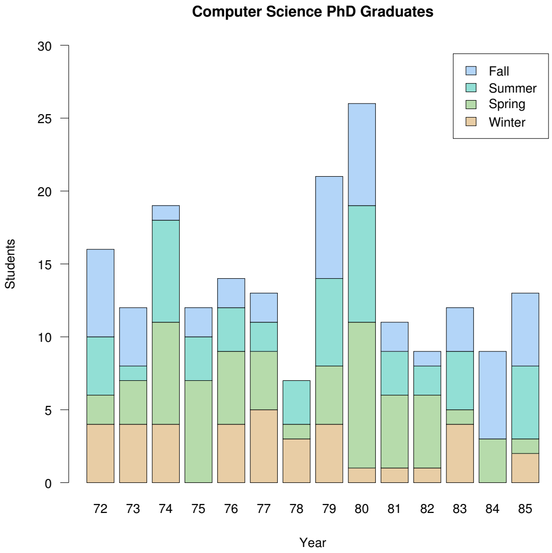


A Six Colour Palette (L = 75, C = 55)



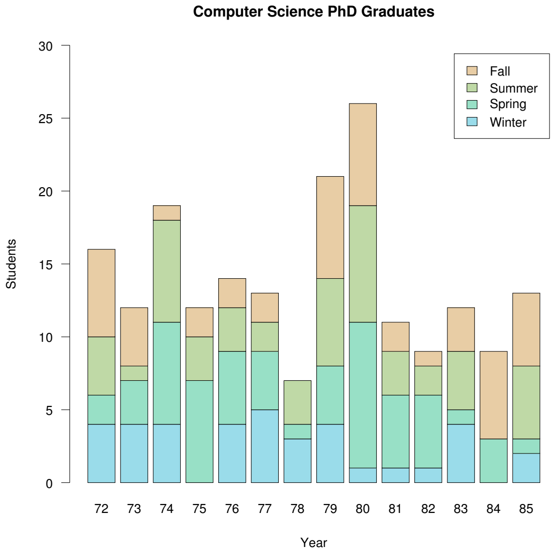
Equispaced colours, full circle.

```
hcl(h = 30 + seq(0, 360, length=5)[1:4], 35, 85)
```



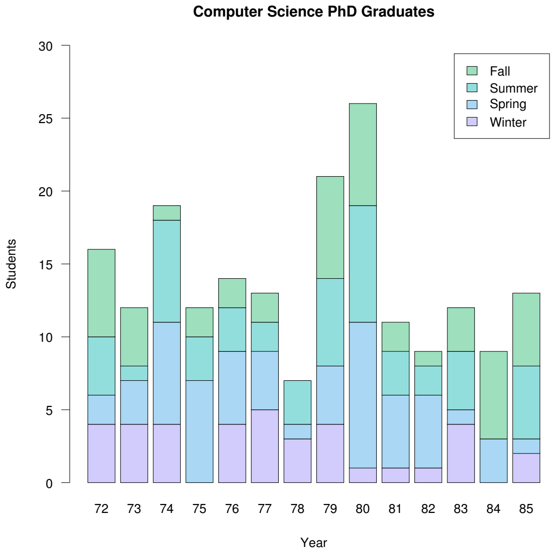
Equispaced colours, half circle.

```
hcl(h = seq(60, 240, by = 60), 35, 85)
```



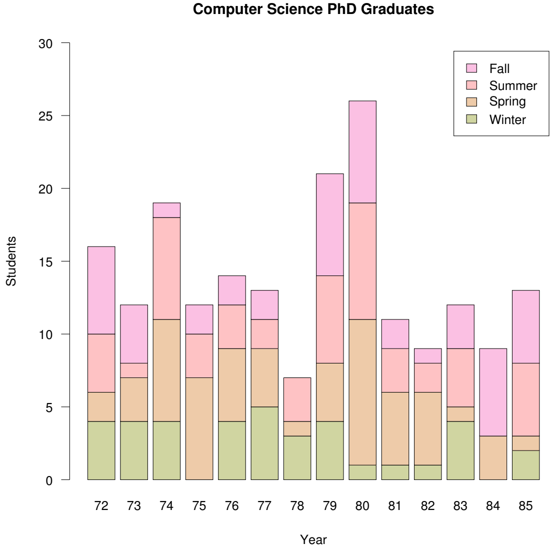
Metaphorical colours.

```
csphd(hcl(h = seq(210, 60,length = 4), 35, 85))
```



Cool colours.

```
hcl(h = seq(120, 0, length = 4) + 150, 35, 85)
```



Warm colours.

```
hcl(h = seq(120, 0, length = 4) - 30, 35, 85)
```

Drawing Smooth Curves

- Given an arbitrary function f and an interval $[a, b]$, I want to have an automatic procedure that will draw the graph of the function over the interval.

- Using the procedure should be as easy as this.

```
    plot(f, a, b)
```

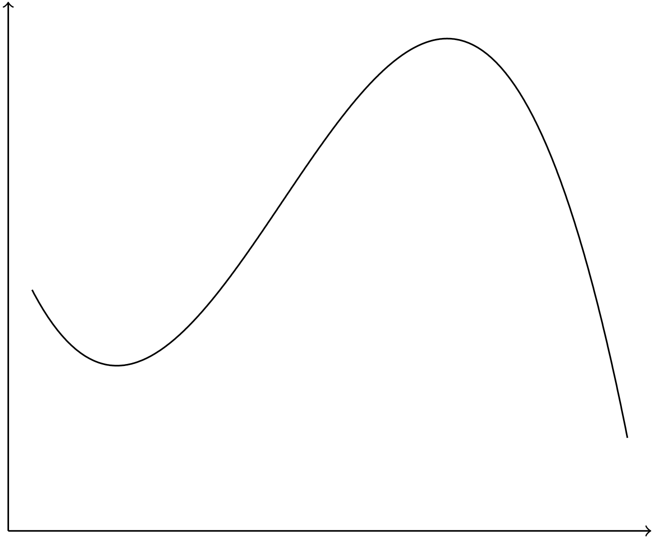
- At this level of generality the problem is “hard” but, for more restricted classes of function, progress can be made.

- This talk will mostly consider smooth (i.e. differentiable) functions but will also mention piecewise-smooth functions.

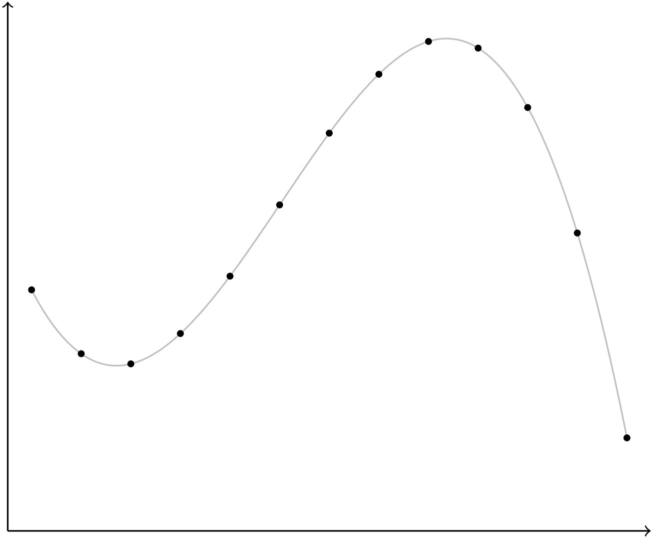
Curves and Polylines

- In computer graphics, a *polyline* is a connected set of line segments.
- A standard method for drawing parametric curves in a vector drawing system uses a polyline approximation based on (many) equally-spaced parameter values.
- Drawing the graph of a function $f(x)$ over an interval $[a,b]$ is an important special case.
- This is the method used in the R function `curve`, which uses 101 equally-spaced points by default.

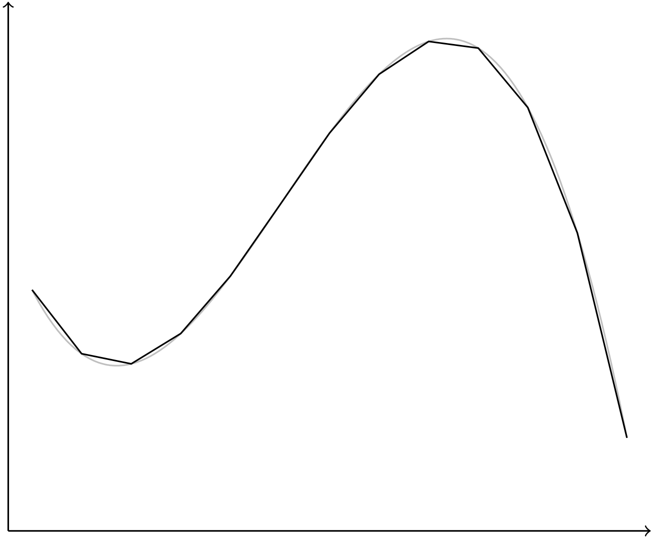
A Smooth Function



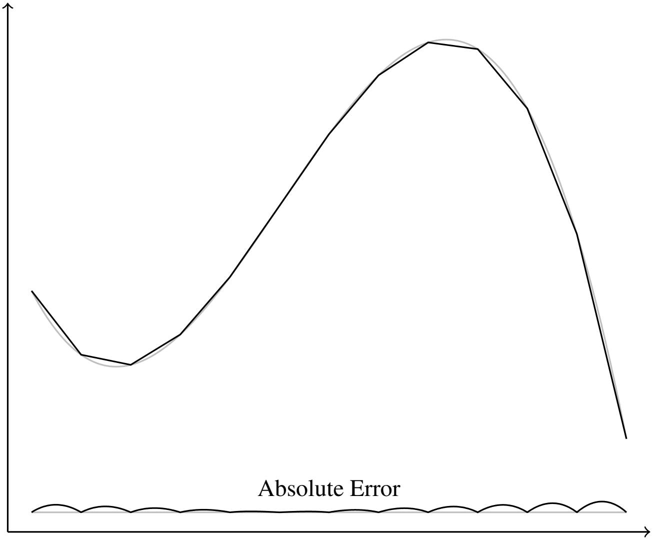
A Smooth Function



A Smooth Function



A Smooth Function



What Makes a Polyline Approximation Good?

- Even when the accuracy of the polyline approximation is good, it may not look “right” because it is clearly not a smooth curve.

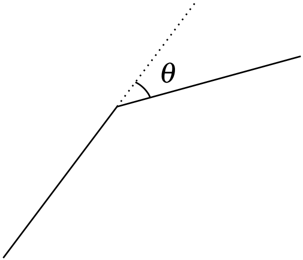
- This is because the visual system detects the “corners” at the joins of the segments making up the polyline.

- This is very much like the edge-detection process built into the visual system.

- The real issue here is one of *visual perception* rather than mathematical accuracy.

Visual Detection of Polyline Joins

- As with many perceptual phenomena, there appears to be a threshold effect for detection of a join between polylines.
- When the change direction, θ, of the polylines is "small" (e.g. $< 3^\circ$), no join is detected and the transition from one segment to the next is seen as "smooth."



Drawing Visually Smooth Polylines

- To draw a polyline approximation to a curve that is visually smooth:
  - Choose points along the curve so that the change of angle from segment to segment is less than the “visually smooth” threshold.
  - Draw the polyline that joins the points.

Mathematical “Formulation”

- Given a smooth function $f : [a,b] \to \mathbb{R}$, let $\phi(x)$ be the angle that the tangent to $f(x)$ at x makes with the x-axis.
- We want to choose $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ so that the change of angle from x_i to x_{i+1} is small.
- I.e., for some suitably small δ the x_i should satisfy

  $$\left|\phi(x_{i+1}) - \phi(x_i)\right| < \delta$$

  for $i = 1, \ldots, n$.
- How can the x_i values be chosen to make this happen?

A Solution

The function

$$h(x) = \int_a^x \left|\phi'(t)\right| dt.$$

is a monotonically non-decreasing function of x and so has a well-defined inverse. Further, if $u < v$,

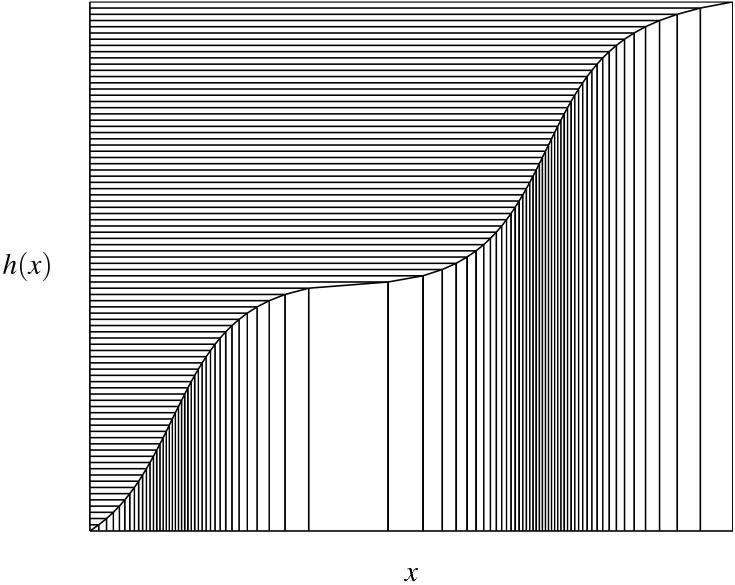
$$|\phi(v) - \phi(u)| = \left|\int_u^v \phi'(t)\, dt\right| \leq \int_u^v \left|\phi'(t)\right| dt = h(v) - h(u)$$

Choose points $h(a) = y_0 < y_1 < y_2 < \cdots < y_n = h(b)$ such that $y_{i+1} - y_i < \delta$ for all i. Set $x_i = h^{-1}(y_i)$ then the points

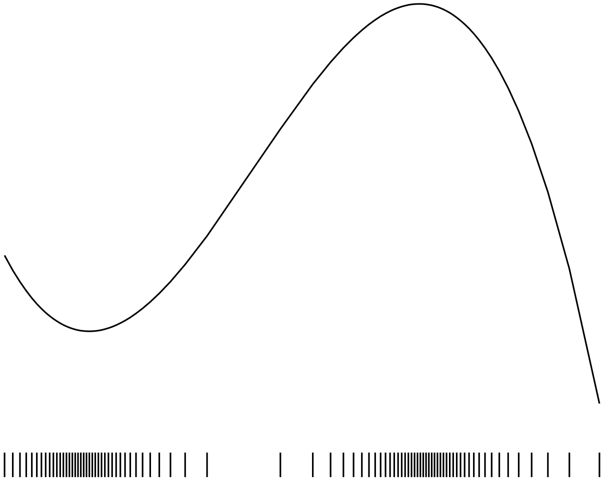
$$\big(x_0, f(x_0)\big), \big(x_1, f(x_2)\big), \ldots, \big(x_n, f(x_n)\big)$$

define the approximating polyline.

Choice of Plotting Ordinates



A Smooth Polyline Approximation (86 Points)



Computer Implementation

- To implement the method in a computer the continuous problem is approximated by its discrete version using a very finely spaced grid of points splitting up the interval from a to b.
  - Derivatives are replaced by differences.
  - Indefinite integrals are replaced by cumulative sums.
- Linear interpolation can be used to compute $h(x)$ and $h^{-1}(u)$.

An R Implementation

```
smoothx =
    function(f, a, b, delta = 2.5, n = 1001) {
        dr = delta * pi / 180
        x = seq(a, b, length = n)
        y = f(x)
        <Compute scalings ax and ay>
        phi = atan2(ay * diff(y), ax * diff(x))
        d.phi = c(0, diff(phi), 0)
        u = cumsum(abs(d.phi) + 1e-3/n)
        if (u[n] - u[1] < dr)
            x[c(1, length(x))]
        else
            approx(u, x,
                   n = ceiling((u[n] - u[1])/dr) + 1)$y
    }
```

Piecewise Continuous Functions

- There are many (useful/interesting) functions that not continuous on their domain or which lie partly outside the plotting window of interest.
- Extending the drawing method outlined previously requires decomposing the domain of a function into intervals where the smooth polyline approximation can be used.
- This means dealing with:
  - Regions where the function is out of the viewing window (clipping).
  - Points where the function is not defined.
  - Simple function discontinuities.
  - Asymptotes.

Houston, We Have a Problem

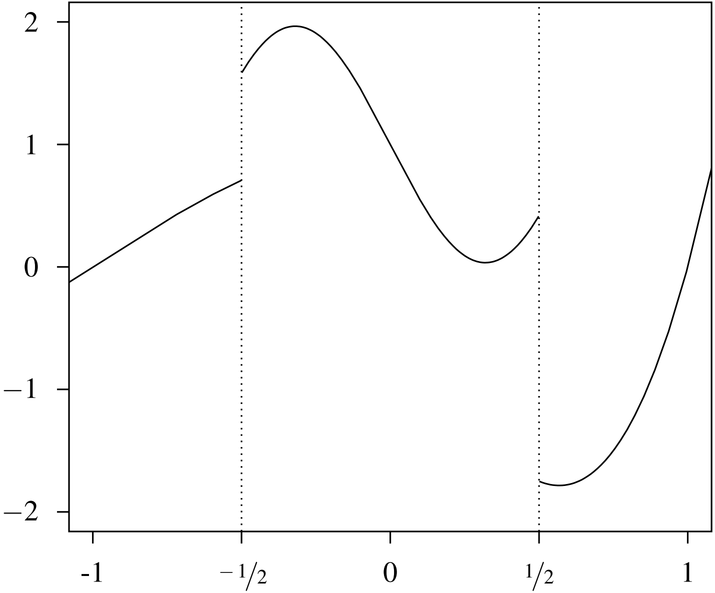
Patent No.: US 6,704,013 B2
Michael E. Hosea, 2004 (Texas Instruments)
FUNCTION GRAPHING METHOD WITH DETECTION OF DISCONTINUITIES

Patent No.: US 7,595,801 B1
Barry M. Cherkas, 2009
COMPLETE FUNCTION GRAPHING SYSTEM AND METHOD

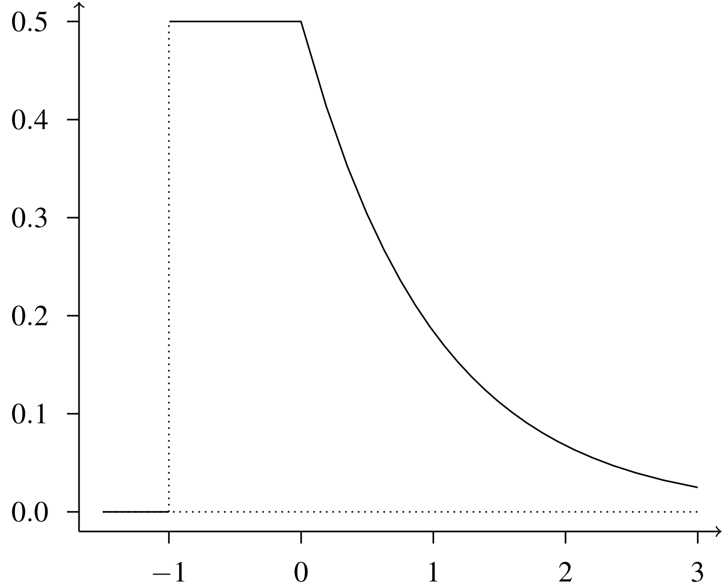
Patent No.: US 7,920,142 B2
Luke Kelly and Jinsong Yu, 2009 (Microsoft)
IDENTIFYING ASYMPTOTES IN APPROXIMATED CURVES AND SURFACES.

These (software) patents make work in this area problematic.

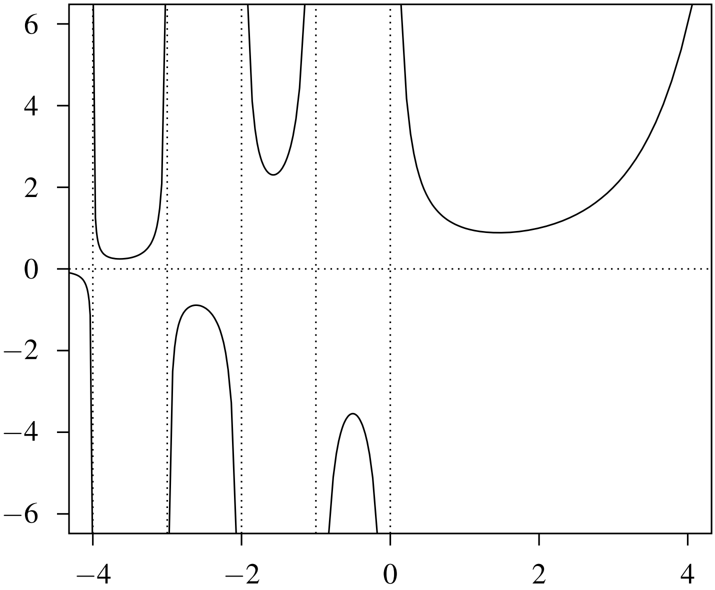
Example: A Discontinuous Function



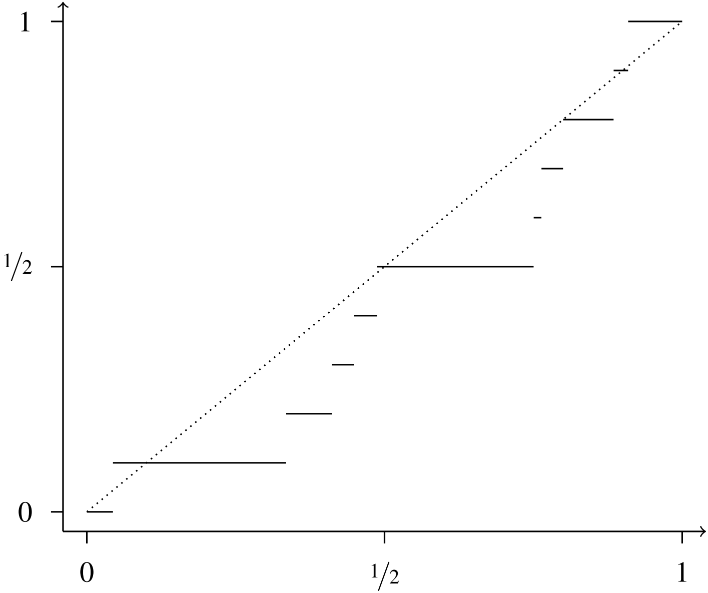
Example: A Density From Ash's *Basic Probability Theory*



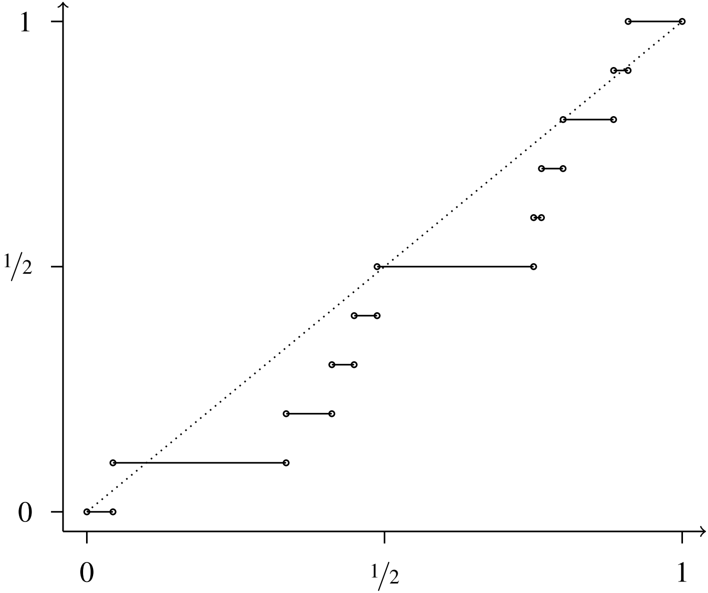
Example: The Gamma Function (A Torture Test)



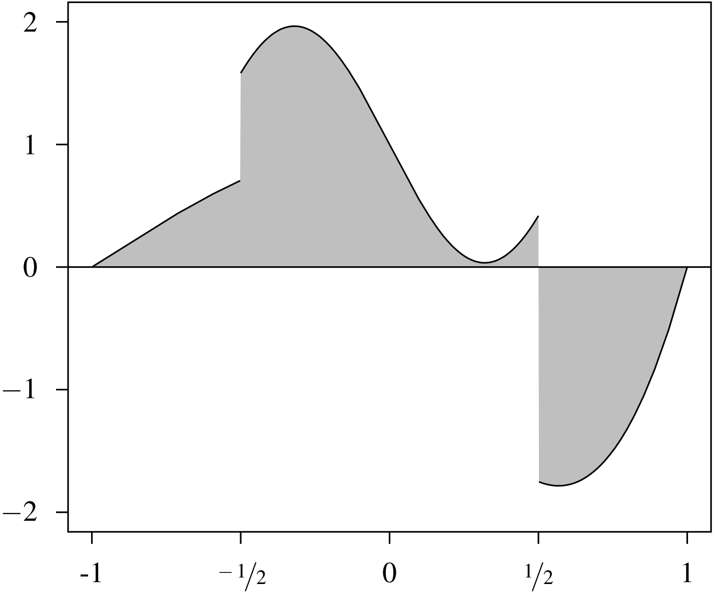
Example: An Empirical CDF



Example: An Empirical CDF



Curves and Area Fills



Summary

- Keeping the constraints and capabilities of human perception in mind can help with the production of software for drawing information (and other) graphics.
- Two examples were presented.
  - Choosing colours for area fills.
  - Drawing smooth curves.