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Finding Geometric Structure in Chaos

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Introduction

A system that changes with time is called a *dynamical system*. The weather, fluctuations in the stock market, lasers and the beat of a human heart are all examples of dynamical systems. Many dynamical systems display long-term unpredictability. A phenomenon known as the *butterfly effect* is responsible: if a butterfly flaps its wings in New Zealand it can cause a hurricane in Florida. Mathematically this is called *sensitive dependence on initial conditions,* and it is a signal that there is chaos in a system.

The problem

We study a famous dynamical system called the Lorenz system, which was invented by meteorologist Edward Lorenz. The equations are a simplification of a model for thermal convection:

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = \rho x - y - xz$$
$$\dot{z} = xy - \beta z$$

Here (x,y,z) represent a point in 3-dimensional space controlled by a force field given by the equations. The Greek letters ρ, σ and β are the parameters of the system. Lorenz discovered chaos for: $\rho = 28$, $\sigma = 10$ and $\beta = 8/3$.

The Lorenz system is a classical example of a chaotic system [1]. It has been studied extensively since it was published in 1963 [2]. Although the equations themselves look relatively simple they cannot be solved explicitly. Tucker proved the existence of chaos in the Lorenz system in 1999 [3]. However this proof only holds for values of ρ up to around 30. From that point an unknown change happens and the system gets a lot more complicated. This means the analytic techniques currently used do not work for larger ρ and much is still unknown about the system:

- How do we find structure in the chaos?
- What are the geometric structures within the system and how do they change with the parameters?

Components of chaos

The basic geometric structures in the Lorenz system are:

Points of equilibrium: O at the origin, p+ and p-

Each of these equilibria are of *saddle type* which means that when nearby points approach it they zoom away vertically before reaching it. This behaviour creates curves and surfaces within the force field, called *manifolds*.

One-dimensional manifolds of p+ and p-

The 1D stable manifolds of p+ and p- are formed by the collection of points that end up at p+ or p- respectively. These manifolds are quite easy to calculate as the points lie on curves through the equilibria.

Two-dimensional manifold of O

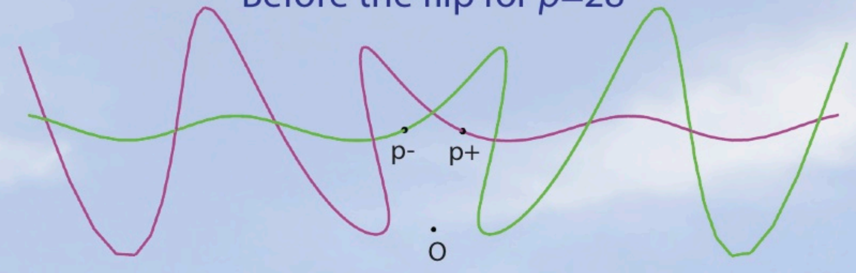
The 2D stable manifold is a surface formed by the collection of points that end up at the origin, the point O=(0,0,0). Calculated using dedicated algorithms, the picture above shows this surface radiating from the origin and spiraling tightly around itself.

The stable manifolds are the key to understanding the structure of the system. They must wrap around each other because they cannot intersect. This means that the curves organise the surfaces and the surfaces organise the 3D space, and so the chaos, in the Lorenz system.

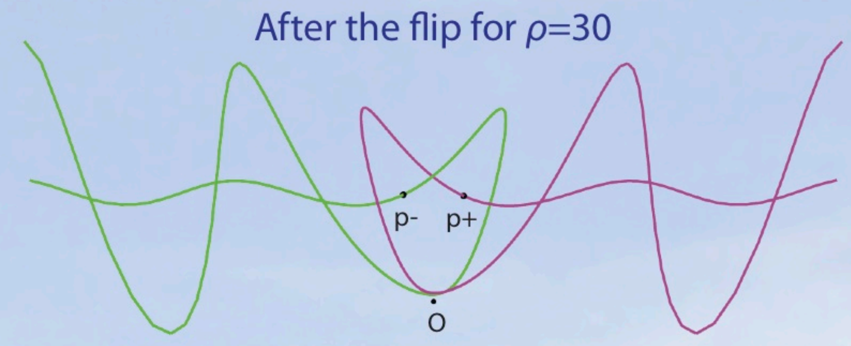
Understanding 'The Flip'

At $\rho \approx 29.7$ the manifolds of p+ and p- undergo a dramatic change. The images below show the manifolds spiraling around each other. The top picture shows the set up for ρ=28. For ρ=30, the bottom picture, we can see that the manifolds now loop around and head back out the way they came in.

Before the flip for ρ=28



After the flip for ρ=30



Notice how one side of the manifold has switched to the other, hence calling it a *flip*. As previously mentioned the curves are wrapped tightly around the surfaces, so what does this flip mean for the 2D manifold of the origin?

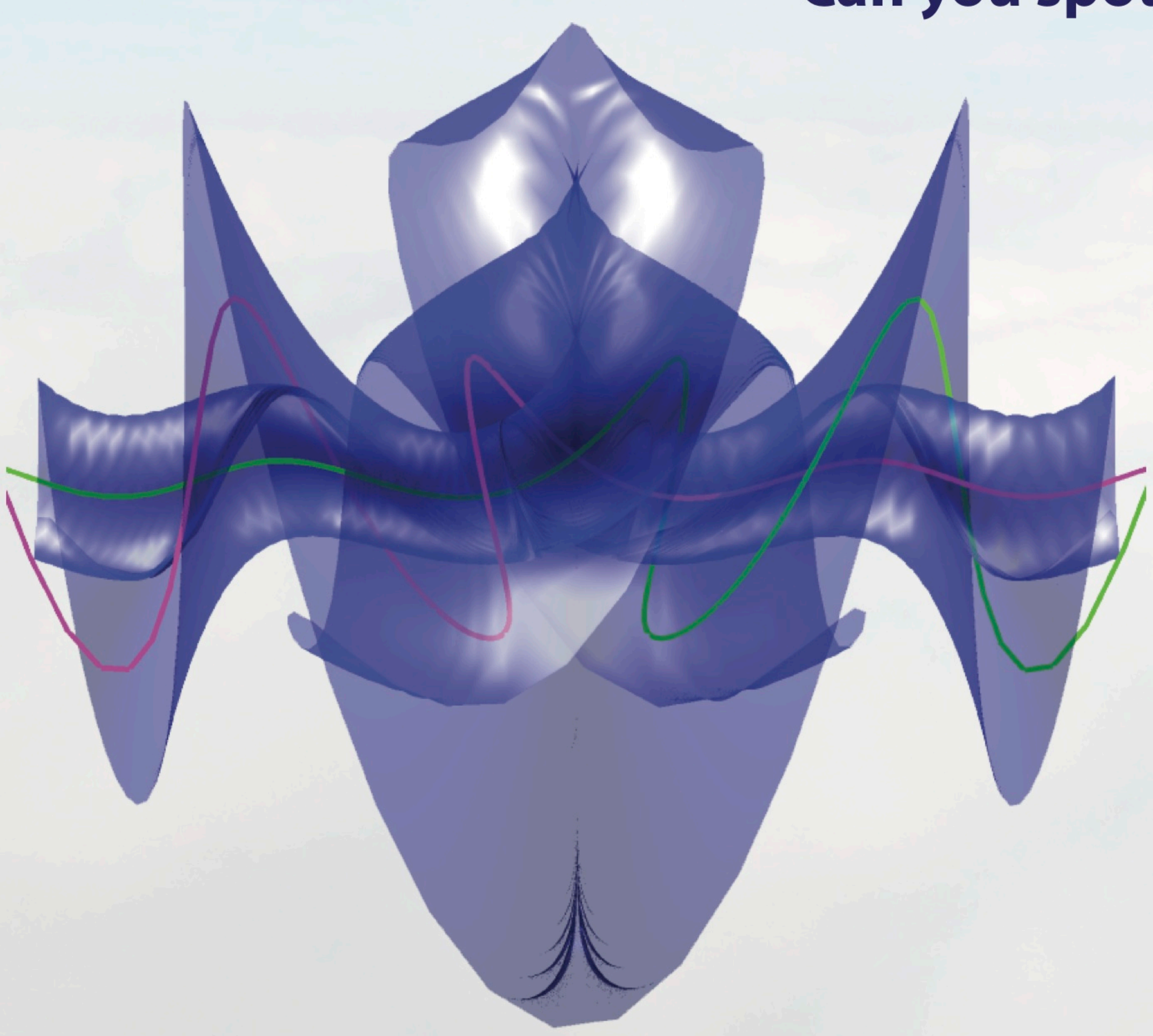
Conclusions

Actually, there is no change in the 2D stable manifold of the origin when the 1D manifolds 'flip'. The structures are intertwined so when the curves move they squeeze in between the folds of the surface and around to the other side. Initial indications suggest that this flip is associated with the change that happens near ρ=30.

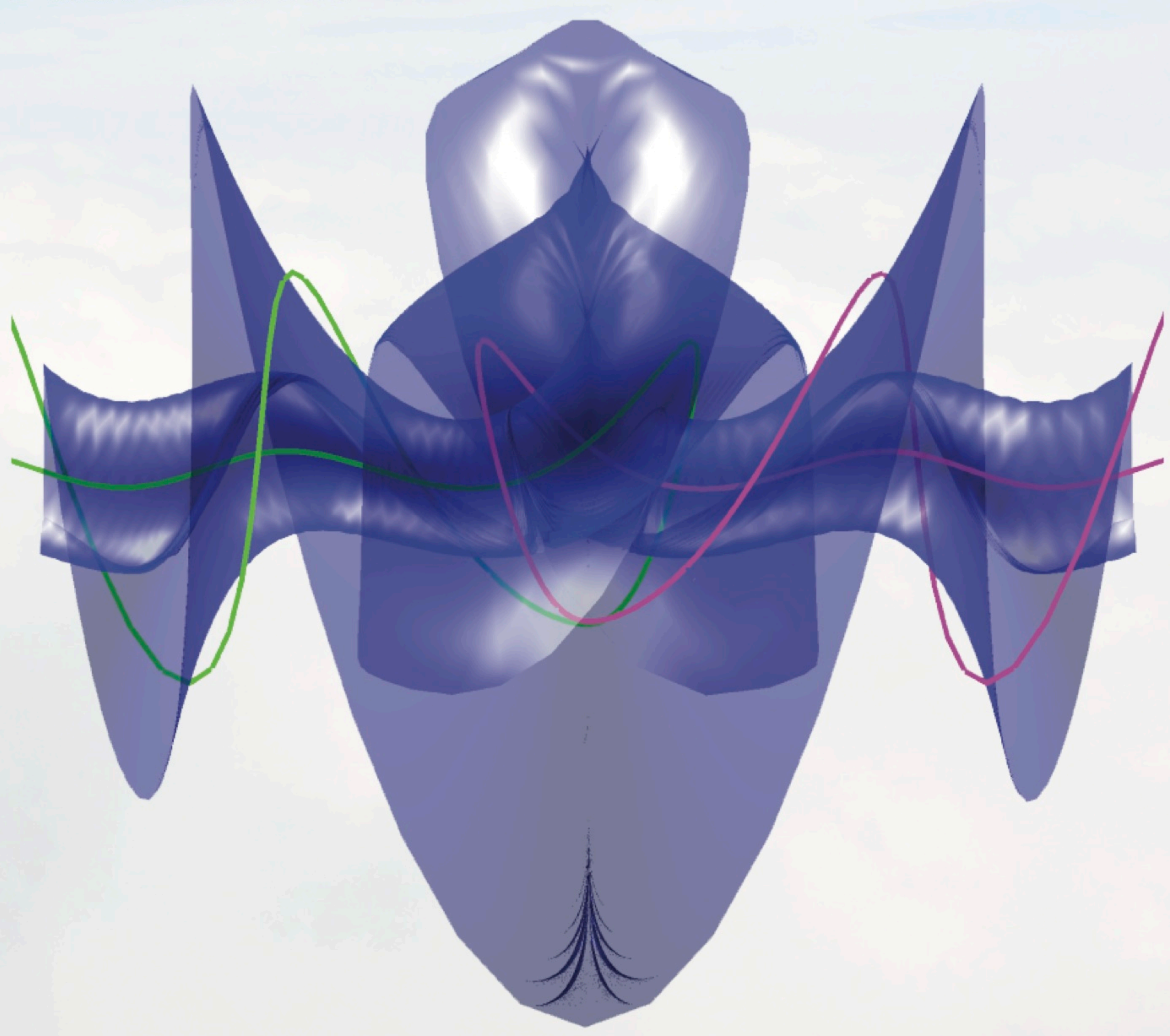
A further 5 flips have been found for ρ values up to 250. This leads to many questions:

- Why do the manifolds of p+ and p- flip in this way?
- Do the flips happen at regular intervals?
- Do they continue to flip around for all values of ρ?
- What does this mean for the 2D manifold and the organisation of chaos in the system?

Can you spot the difference?



Before the flip for ρ=28



References
[1] Gleick, J., 1987, Chaos: Making a New Science, Viking Penguin. [2] Lorenz, Edward N., 1963, Deterministic Nonperiodic Flow. J. Atmos. Sci., 20, 130–141. [3] Tucker, W., 1999, The lorenz attractor exists. C. R. Acad. Sci. Paris, 328, Serie I: 1197– 1202.