1. Suppose we have a model with a continuous response $y$ and two factors $A$ and $B$, each having 3 levels. Each factor level combination is measured twice. Carefully explain how R constructs the model matrix (i.e. the matrix of dummy variables) for each of the models
   
   (a) $y \sim A \times B$;  
   (b) $y \sim A + B$.

   In each case, illustrate your answer with a numerical example. You may find the R function `model.matrix` useful. [10 marks]

2. Draw a contour plot of the logistic regression log-likelihood for the CHD data. Mark the position of the maximum on the plot. [10 marks]

3. Suppose we have a contingency table with $m$ cells, and we want to fit a model to the table.
   
   (a) The “multinomial” model models the cell counts $y_1, \ldots, y_m$ as a multinomial distribution, with parameters $n = \sum_{i=1}^{m} y_i$ and $\pi_i, \ i = \ldots, m$. In terms of these parameters, the log-likelihood for this model is

   $$l_M = \sum_{i=1}^{m} y_i \log(\pi_i).$$

   Suppose we parameterise the probabilities $\pi_i$ with $m - 1$ parameters $\alpha_2, \ldots, \alpha_m$, by setting

   $$\pi_1 = \frac{1}{1 + \sum_{i=2}^{m} e^{\alpha_i}},$$

   $$\pi_i = \frac{e^{\alpha_i}}{1 + \sum_{i=2}^{m} e^{\alpha_i}}, \ i = 2, \ldots, m.$$  

   Show that, in terms of the new parameters, the log-likelihood is

   $$l_M(\alpha_2, \ldots, \alpha_m) = \sum_{i=2}^{m} y_i \alpha_i - n \log \left( 1 + \sum_{i=2}^{m} e^{\alpha_i} \right).$$

   [4 marks]
(b) Calculate the derivative of \( l_M \) and hence show that the MLE’s of \( \alpha_2, \ldots, \alpha_m \) are the solution to the equations

\[
y_i = n \frac{e^{\alpha_i}}{\left(1 + \sum_{i=2}^{m} e^{\alpha_i}\right)}, \quad i = 2, \ldots, m.
\]

(1)

[4 marks] Hint: Recall that the derivatives are zero at the maximum.

(c) The “Poisson” model models the cell counts \( y_1, \ldots, y_m \) as independent Poisson distributions, with means \( \mu_i, \quad i = 1, \ldots, m \). In terms of these parameters, the log-likelihood for this model is

\[
l_P = \sum_{i=1}^{m} \{ y_i \log(\mu_i) - \mu_i \}.
\]

Suppose we re-parameterise the means with \( m \) parameters \( \mu, \alpha_2, \ldots, \alpha_m \), by setting

\[
\begin{align*}
\mu_1 &= \exp(\mu), \\
\mu_i &= \exp(\mu + \alpha_i), \quad i = 2, \ldots, m.
\end{align*}
\]

Show that, in terms of the new parameters, the log-likelihood can be written

\[
l_P(\mu, \alpha_2, \ldots, \alpha_m) = n\mu - e^\mu + \sum_{i=2}^{m} (y_i\alpha_i - e^{\mu+\alpha_i}).
\]

[4 marks]

(d) Calculate the derivative of \( l_P \) and hence show that the MLE’s of \( \alpha_2, \ldots, \alpha_m \) for the Poisson model are the same as those for the multinomial model.[4 marks]

(e) Use the “death by falling” example to verify numerically that the coefficients calculated in R using the Poisson model are the same as the solutions to (1). [4 marks]