1. (a) Consider each of the following sets of vectors in $R^4$ where $a$, $b$ and $c$ represent any real numbers.

All vectors of the form:

$$
\begin{pmatrix}
    a \\
    b \\
    a + b \\
    1
\end{pmatrix}
$$

(ii)

$$
\begin{pmatrix}
    a \\
    b \\
    a + b \\
    0
\end{pmatrix}
$$

(iii)

$$
\begin{pmatrix}
    a \\
    b \\
    c \\
    a + b - 2c
\end{pmatrix}
$$

In each case, determine whether the set of vectors represents a subspace of $R^4$ (you need to justify your answer).

(b) Consider the subspace of $R^4$ defined as all vectors of the form

$$
\begin{pmatrix}
    a \\
    a + b \\
    a - b \\
    -2b
\end{pmatrix}
$$

What is the dimension of this subspace? Give a set of vectors that form a basis for this subspace.

(c) Consider a random vector $X \sim N_3(\mu_X, \Sigma_X)$:

$$
X = \begin{bmatrix}
    X_1 \\
    X_2 \\
    X_3
\end{bmatrix}, \quad \mu_X = \begin{bmatrix}
    \mu_1 \\
    \mu_2 \\
    \mu_3
\end{bmatrix}, \quad \Sigma_X = \begin{bmatrix}
    2 & 1 & 2 \\
    1 & 2 & -1 \\
    2 & -1 & 3
\end{bmatrix}
$$

i. Find a matrix $A$ such that if we let $W = AX$ then:

$$
\mu_W = \begin{bmatrix}
    \mu_1 - \mu_2 \\
    \mu_2 + 2\mu_3 \\
    3\mu_1 - 2\mu_2 - \mu_3
\end{bmatrix}
$$

ii. Find the covariance matrix for $W$, i.e. $\Sigma_W$.

(d) The Best Linear Unbiased Estimate (BLUE) of a parameter is defined as the estimator that has the smallest variance among the set of all unbiased linear estimators. Note a linear estimator is simply any estimator which is a linear combination of the data, i.e. can be written as $a^t Y$.

Given the linear model,

$$
Y = X\beta + \epsilon \quad \text{where} \quad \beta \sim N(0, \sigma^2 I_n)
$$

and the least squares estimates of $\beta$

$$
\hat{\beta} = (X'X)^{-1} X'Y,
$$

consider estimating a linear combination of the parameters $a^t \beta$. We can show that among all possible linear unbiased estimators of $a^t \beta$, $a^t \hat{\beta}$ has the smallest variance using the following steps.
i. First, consider any linear estimator $b'Y$ and show:
   A. $b'Y$ is unbiased for $a'\beta$ if $b'X = a'$.
   B. $\text{Var}(b'Y) = \sigma^2 b'b$.

ii. Now consider the estimator based on the least squares estimates $a'\hat{\beta}$. Show the following:
   A. $a'\hat{\beta}$ is unbiased for $a'\beta$,
   B. $\text{Var}(a'\hat{\beta}) = \sigma^2 a'(X'X)^{-1}a$

iii. Now set $b^* = X(X'X)^{-1}a$. For any $b$

$$ (b - b^*)' (b - b^*) \geq 0 $$

and the equality holds only if $b = b^*$. Show that

$$ \sigma^2 (b - b^*)' (b - b^*) = \text{Var}(b'Y) - \text{Var}(b^*Y) $$

which means $\text{Var}(b'Y) \geq \text{Var}(b^*Y)$ and the equality holds only if $b = b^*$.

Hint: expand the left hand side and simplify using the results in (a) and (b).