Extended Exercise on Sample Size and Power

This is an extended exercise in which we introduce some ideas about determining the sample size for hypotheisis testing and also introduce and work with the idea of the ***power*** of a test.

Suppose that we want to test $H_0 : p = p_0$. We will use the large sample Normal approximation to the distribution of $\widehat{P}$ itself to find *P-values* . Let $\text{sd}_0 = \sqrt{p_0(1-p_0)/n}$.

We first explore the one sided test $H_0 : p = p_0$ versus $H_1 : p > p_0$.

(a) Show that we will reject H_0 at the 5% level of significance if $\widehat{p}$ is greater than $p_0 + 1.645\ \text{sd}_0$.

Let us now apply this to testing $H_0 : p = 0.5$ versus $H_1 : p > 0.5$ when the sample size is $n = 100$.

(b) For what values of $\widehat{p}$ do we reject H_0?

(c) What is the probability of rejecting $H_0 : p = 0.5$ if the true value of p is 0.6?

The probability of rejecting $H_0 : p = 0.5$ in favor of $H_1 : p > 0.5$ when the true value of p is 0.6 is called the ***power*** of the test when $p = 0.6$. Naturally, when the true value differs from the hypothesized value, we want the power of the test (probability of rejecting H_0) to be as large as possible. We will now lead you through a power calculation for the two-sided test $H_0 : p = p_0$ versus $H_1 : p \neq p_0$.

(d) Show that we reject H_0 at the 5% level of significance if $\widehat{p} > p_0 + 1.96\ \text{sd}_0$ or $\widehat{p} < p_0 - 1.96\ \text{sd}_0$.

Let us now apply this to testing $H_0 : p = 0.5$ versus $H_1 : p \neq 0.5$ when $n = 100$.

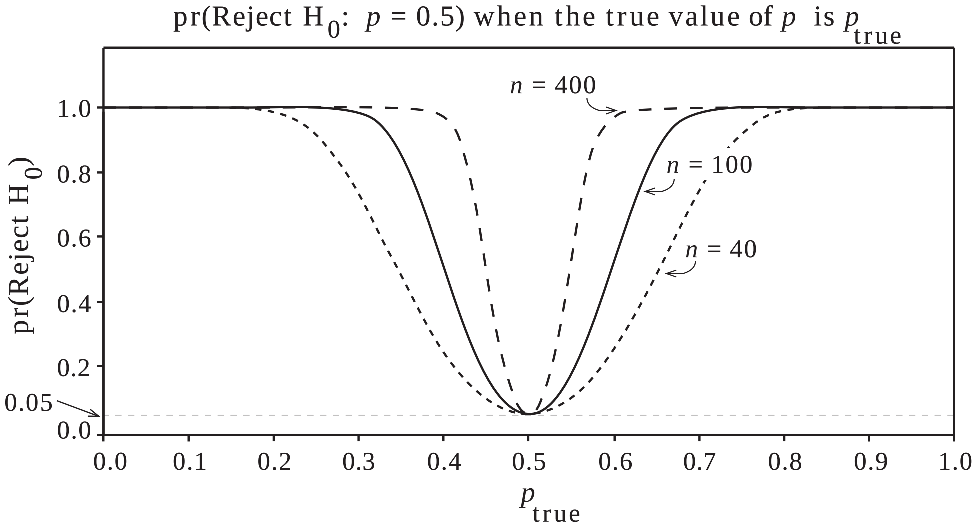
(e) For what values of $\widehat{p}$ do we reject H_0.

(f) What is the *power* of the test if $p = 0.6$? In other words, what is the probability of rejecting $H_0 : p = 0.5$ if the true value of p is 0.6?

Below, we graphed[1] pr(Reject H_0 when the true value of p is p_{true}) against p_{true} for a sample of size $n = 100$ (solid curve). We repeated these calculations for a sample of size $n = 400$ (dashed line) and $n = 40$ (dotted line).

(g) What do these plots tell you?

[1] This is called the ***power function*** for the test.



Finally, we will lead you through a sample size calculation. Let $\mathrm{sd}_T = \sqrt{p_T(1-p_T)/n}$.

(h) Show that a one-sided test of $H_0: p = p_0$ versus $H_1: p > p_0$ performed at the 5% level of significance has power 80% (i.e. the probability of rejecting H_0 is 0.8) if the true value of p is $p_T = p_0 + 1.645\ \mathrm{sd}_0 + 0.8416\ \mathrm{sd}_T$.

(i) Suppose that $p_0 = 0.5$ and the true value of p is $p_T = 0.6$. Using the equation in (h), find n so that a 5% significance level test has an 80% chance of rejecting H_0.