Chapter 5 Discrete Random Variables

Exercises for Section 5.2

1. Probabilities must lie between 0 and 1 and add to 1. The value 1.10 is clearly incorrect for a probability. If it is changed to 0.11 then the set of probabilities add to 1.

2. $\text{pr}(X = 1) = \frac{176}{200} = 0.88$, $\text{pr}(X = 2) = \frac{22}{200} = 0.11$, etc. Placing these into a table, we get the following.

x	1	2	3
pr(x)	0.88	0.11	0.01

3. We have, using independence, $\text{pr}(BBB) = \text{pr}(B) \times \text{pr}(B) \times \text{pr}(B) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$. Every other outcome has the same probability. (This is the same probability distribution as in Example 4.4.6(b).) $X = 0$ corresponds to one outcome (BBB) and thus has probability $\frac{1}{8}$. $X = 1$ corresponds to three outcomes (GBB, BGB, and BBG) and thus has probability $\frac{3}{8}$, and so on. We obtain the following.

x	0	1	2	3
pr(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

4. $\text{pr}(4 < X < 8) = \text{pr}(X = 5) + \text{pr}(X = 6) + \text{pr}(X = 7) = 0.844$ or $\text{pr}(4 < X < 8) = \text{pr}(X \leq 7) - \text{pr}(x \leq 4) = 0.882 - 0.038 = 0.844$

5. **(a)** The cumulative probabilities are given below.

x	3	4	5	6	7	8	9	10	11	12	13
$\text{pr}(X = x)$	.07	.01	.09	.01	.16	.25	.20	.03	.02	.11	.05
$\text{pr}(X \leq x)$	.07	.08	.17	.18	.34	.59	.79	.82	.84	.95	1.0

(b) $\text{pr}(X \leq 5) = \text{pr}(3) + \text{pr}(4) + \text{pr}(5) = .07 + .01 + .09 = 0.17$. We also read the value directly off the cumulative probabilities row.

(c) $\text{pr}(X > 9) = \text{pr}(10) + \text{pr}(11) + \text{pr}(12) + \text{pr}(13) = .03 + .02 + .11 + .05 = 0.21$, or, using the cumulative probabilities row, $\text{pr}(X > 9) = 1 - \text{pr}(x \leq 9) = 1 - 0.79 = 0.21$.

(d) $\text{pr}(X \geq 9) = \text{pr}(9) + \ldots + \text{pr}(13) = 0.41$, or, using the cumulative probabilities row, $\text{pr}(X \geq 9) = 1 - \text{pr}(x \leq 8) = 1 - 0.59 = 0.41$.

(e) $\text{pr}(X < 12) = \text{pr}(3) + \ldots + \text{pr}(11) = 0.84$, or using the cumulative probabilities table, $\text{pr}(X < 12) = \text{pr}(X \leq 11) = 0.84$.

(f) $\text{pr}(5 \leq X \leq 9) = \text{pr}(5) + \ldots + \text{pr}(9) = 0.71$, or using the cumulative probabilities table, $\text{pr}(5 \leq X \leq 9) = \text{pr}(X \leq 9) - \text{pr}(X \leq 4) = 0.79 - 0.08 = 0.41$.

(g) $\text{pr}(4 < X < 11) = \text{pr}(5) + \ldots + \text{pr}(10) = 0.74$, or using the cumulative probabilities table, $\text{pr}(4 < X < 11) = \text{pr}(X \leq 10) - \text{pr}(X \leq 4) = 0.82 - 0.08 = 0.74$.

(h) and **(i)** The probabilities for the values of x listed in the table add to 1 so these take up all the possible values. Everything outside this has probability 0. Thus $\text{pr}(X = 14) = 0$ and $\text{pr}(X < 3) = 0$.

6. Arguing exactly as in the Case Study, $0.9^3 = 0.729$. As $\text{pr}(X \leq 6) = 0.469$ and $\text{pr}(X \leq 7) = 0.522$, she should try seven times.

Exercises for Section 5.3

1. **(a)** 0.2668. **(b)** $1 - \text{pr}(X \leq 3) = 0.3504$. **(c)** $1 - \text{pr}(X \leq 6) = 0.01059$.
(d) $\text{pr}(X \leq 6) = 0.9894$. **(e)** 0.1493. **(f)** $\text{pr}(X \leq 7) - \text{pr}(X \leq 3) = 0.3488$.
(g) $\text{pr}(X \leq 7) - \text{pr}(X \leq 3) = 0.3488$. **(h)** 0 as X only takes values $0, 1, \ldots, 10$.
(i) $\text{pr}(X \leq 10) = 1$.

2. **(a)** Here the 20% refers to a conceptual population rather an actual one, so that the coin-tossing model is a candidate. The urn model does not apply, as the 10 chosen cars are not a simple random sample of all the cars parking. You need to have the probability of overstaying remaining constant at 0.2. Here $n = 10$ and $p = 0.2$. The arrival of each car is like a binomial trial. You can use Binomial$(n = 10, p = 0.2)$.

(b) Urn model. The 50 cars watched must be a simple random sample of the cars. Here $N = 10,000$; M, the total number of overstayers, is unknown; and $n = 50$. You can approximate by Binomial$(n = 50, p)$, where $p = \frac{M}{N}$, as $\frac{n}{N} < 0.1$. You may be able to use $p = 0.2$ from part (a).

(c) Urn model. The 50 people dialed must be a simple random sample of subscribers. Here $N = 7400$, $M = 2730$, and $n = 50$. You can approximate by Binomial$(n = 50, p = \frac{M}{N} = 0.3689)$ as $\frac{n}{N} < 0.1$.

(d) Urn model. Here N and M are large unknown numbers, $\frac{M}{N} = 0.45$, and $n = 100$. You can approximate by Binomial$(n = 100, p = 0.45)$ as you can expect $\frac{n}{N} < 0.1$.

(e) Urn model. Here $N = 100$, $M = 12$ and $n = 7$. You can approximate by Binomial$(n = 100, p = \frac{M}{N} = 0.12)$ as $\frac{n}{N} = \frac{7}{100} < 0.1$.

(f) Urn model. Here N and M are large unknown numbers, $\frac{M}{N} = 0.64$, and $n = 50$. You can approximate by Binomial$(n = 50, p = 0.64)$ as you can expect $\frac{n}{N} < 0.1$.

(g) Urn model. Here $N = 188$, $M = 99$, and $n = 30$. You cannot use the Binomial as $\frac{n}{N} > 0.1$.

(h) Coin-tossing model. Here $n = 10$ and $p = \frac{1}{6}$. Can use Binomial$(n = 10, p = \frac{1}{6})$

(i) Urn model. Here $N = 52$, $M = 4$, and $n = 7$. You need random shuffling of the pack before dealing. You cannot use the Binomial as $\frac{n}{N} > 0.1$.

(j) Urn model. Here N and M are large unknown numbers, $\frac{M}{N} = 0.1$ and $n = 30$. You can approximate by Binomial$(n = 30, p = 0.1)$ as you can expect $\frac{n}{N} < 0.1$. Either model is a candidate. For an urn model, the 30% must refer to the actual population sampled with M and N unknown and $n = 20$. For the coin-tossing model, you must have p constant. Since you can expect $\frac{n}{N} < 0.1$, either model will lead to Binomial$(n = 20, p = 0.3)$.

(k) Either model is a candidate, both leading to Binomial$(n = 20, p = 0.3)$. For example, urn model applies if we think of the 20 bearings being sampled from the existing population of bearings of which 30% will function for a year of continuous use. If, however, we think in terms of the bearings being random items from a continuous manufacturing process producing items with the "30% will function" being stable (or constant) over time, the coin-tossing model applies.

(l) Like (a), either model is a candidate. The urn model, however, will apply only if the 50 patients can be regarded as a simple random sample of patients, which is probably not the case. The coin-tossing model is dubious as p may not be constant.

Exercises for Section 5.4.1

1. $\mathrm{E}(X) = \sum x\,\mathrm{pr}(x) = 2 \times 0.2 + 3 \times 0.1 + 5 \times 0.3 + 7 \times 0.4 = 5.0.$

2. $\mathrm{E}(X) = \sum x\,\mathrm{pr}(x) = 0 \times 0.49 + 1 \times 0.42 + 2 \times 0.09 = 0.6 \quad (= n\,p).$

Exercises for Section 5.4.2

1. $\mathrm{E}(X-\mu)^2\mathrm{pr}(x) = (2-5)^2 \times 0.2 + (3-5)^2 \times 0.1 + (5-5)^2 \times 0.3 + (7-5)^2 \times 0.4 = 3.8.$
$\mathrm{sd}(X) = \sqrt{3.8} = 1.9494.$

2. $\mathrm{E}(X-\mu)^2\mathrm{pr}(x) = (0-1.25)^2 \times \frac{1}{8} + (1-1.25)^2 \times \frac{5}{8} + (2-1.25)^2 \times \frac{1}{8} + (3-1.25)^2 \times \frac{1}{8}$
$= 0.6875, \qquad \mathrm{sd}(X) = \sqrt{0.6875} = 0.8292.$

3. $\mathrm{E}(X-\mu)^2\mathrm{pr}(x) = (0-0.6)^2 \times 0.49 + (1-0.6)^2 \times 0.42 + (2-0.6)^2 \times 0.09 = 0.42.$
$\mathrm{sd}(X) = \sqrt{0.42} = 0.6481.$

Exercises for Section 5.4.3

(a) $\mathrm{E}(2X) = 2\,\mathrm{E}(X) = 6, \quad \mathrm{sd}(2X) = 2\,\mathrm{sd}(X) = 4.$

(b) $\mathrm{E}(4+X) = 4 + \mathrm{E}(X) = 7, \quad \mathrm{sd}(4+X) = \mathrm{sd}(X) = 2.$

(c) As for (b), since $\mathrm{E}(X+4) = \mathrm{E}(4+X)$ and $\mathrm{sd}(X+4) = \mathrm{sd}(4+X)$.

(d) $\mathrm{E}(3X+2) = 3\,\mathrm{E}(X) + 2 = 11, \quad \mathrm{sd}(3X+2) = 3\,\mathrm{sd}(X) = 6.$

(e) $\mathrm{E}(4+5X) = 4 + 5\,\mathrm{E}(X) = 19, \quad \mathrm{sd}(4+5X) = 5\,\mathrm{sd}(X) = 10.$

(f) $\mathrm{E}(-5X) = -5\,\mathrm{E}(X) = -15, \quad \mathrm{sd}(-5X) = 5\,\mathrm{sd}(X) = 10.$

(g) $\mathrm{E}(-5X+4) = -5\,\mathrm{E}(X) + 4 = -11, \quad \mathrm{sd}(-5X+4) = 5\,\mathrm{sd}(X) = 10.$

(h) As for (g).

(i) $\mathrm{E}(-7X-9) = -7\,\mathrm{E}(X) - 9 = -30, \quad \mathrm{sd}(-7X-9) = 7\,\mathrm{sd}(X) = 14.$

Review Exercises 5

1. In the following "N/A" is used when neither model is applicable.

(a) Coin-tossing model. $X_1 \sim \text{Binomial}(n = 20, p = 0.2)$.

(b) Urn model with $N = 1000$, $M = 100$, and $n = 20$. You can use the Binomial approximation, $X_2 \sim \text{Binomial}(n = 20, p = 0.1)$ as $\frac{n}{N} < 0.1$.

(c) N/A.

(d) Coin-tossing model. $X_4 \sim \text{Binomial}(n = 120, p = 0.6)$.

(e) Urn model with $N = 120$, $M = 70$ and $n = 10$. You can use the Binomial approximation, $X_5 \sim \text{Binomial}(n = 10, p = \frac{7}{12})$, as $\frac{n}{N} < 0.1$.

(f) Urn model with $N = 20$, $M = 9$ and $n = 15$. You cannot use a Binomial approximation as $\frac{n}{N} > 0.1$.

(g) Coin-tossing model. $X_7 \sim \text{Binomial}(n = 12, p = \frac{1}{6})$.

(h) N/A. (Not the number of "heads" in a fixed number of "tosses".)

(i) Coin-tossing model. $X_9 \sim \text{Binomial}(n = 12, p = \frac{1}{36})$.

(j) Urn model with $N = 98$, $M = 44$, and $n = 7$. You can use the Binomial approximation, $X_{10} \sim \text{Binomial}(n = 7, p = \frac{44}{98})$, as $\frac{n}{N} < 0.1$.

3. **(a)** $p = \dfrac{\text{sampled area}}{\text{population area}} = \frac{20\times100\times100}{2000\times2000} = \frac{1}{20}$.

(b) The 420 animals may be regarded as 420 independent Binomial trials each with probability of success, where success means "found in the sample area"and failure means "found outside the sample area." The four assumptions are satisfied because the animals (trials) are independent.
$X \sim \text{Binomial}(n = 400, p = \frac{1}{20})$.

(c) For a single plot $p = \frac{100\times100}{2000\times2000} = \frac{1}{400}$ and $W \sim \text{Binomial}(n = 420, p = \frac{1}{400})$.

(d) Any two of the following.

(i) Animals tend to exhibit social tendencies and so are not generally independent.

(ii) Animals do not move randomly but usually have well-defined territories or "home ranges."

(iii) The presence of observers may disturb the animals so that they move out of the area.

(iv) Some animals may be missed. Deer, for example, are very hard to spot.

5. **(a)** $L \sim \text{Binomial}(n = 160, p = \frac{120}{80{,}000})$.

(b) $\text{E}(L) = np = 0.24$ (about one every four years).

(c) Each ship has the same probability of being lost and ships are lost or not lost independently. This is probably not true, but it may be a reasonable approximation.

7. **(a)** $\text{Binomial}(n = 12, p = 0.18)$.

(b) pr(*No failures*) = pr($X = 0$) = 0.09242.

9. If X = number of rods that perform satisfactorily, you assume that $X \sim$ Binomial($n = 10, p = 0.80$). Then pr($X \le 4$) = 0.006369.

11. **(a)** The value -0.39 is clearly in error (probabilities cannot be negative). We replace it with $1 - (0.23 + 0.18 + 0.17 + 0.13) = 0.29$.

(b) pr($X \ge 1$) = $0.18 + 0.17 + 0.13 = 0.48$.

(c) pr($X \le 0$) = $0.23 + 0.29 = 0.52$.

(d) $\mathrm{E}(X) = \sum x\,\mathrm{pr}(x) = (-3) \times 0.23 + 0 \times 0.29 + 1 \times 0.18 + 3 \times 0.17 + 8 \times 0.13 = 1.04$,
$\mathrm{E}[(X-\mu)^2] = \sum (x-\mu)^2 \mathrm{pr}(x) = (-3 - 1.04)^2 \times 0.23 + (0 - 1.04)^2 \times 0.29.$
$+ (1-1.04)^2 \times 0.18 + (3-1.04)^2 \times 0.17 + (8-1.04)^2 \times 0.13 = 11.0184.$
$\mathrm{sd}(X) = \sqrt{11.0184} = 3.32$.

13. **(a)** In the table, $a(b) = a \times 10^b$.

x	0	1.61	7.85	25.90	34.29	224.13	558.00
pr(x)	0.9333	6.511(−2)	1.190(−3)	2.381(−4)	9.524(−5)	1.720(−5)	1.190(−5)

(b) $\mathrm{E}(X) = \sum x\mathrm{pr}(x)$ is most quickly calculated by noting that here $\sum x\ \mathrm{pr}(x) = \sum \frac{x \times n(x)}{3{,}780{,}000}$, where $n(x)$ is the number of vouchers with value x. Thus $\mathrm{E}(X) = \frac{1.61 \times 246130 + 7.85 \times 4500 + ... + 558.00 \times 45}{3{,}780{,}000} = 0.1341$, or 13.41 cents.

(c) We calculate $\mathrm{E}[(X-\mu)^2] = \sum (x-\mu)^2 \mathrm{pr}(x) = \frac{\sum (x-\mu)^2 \times n(x)}{3{,}780{,}000}$
$= \frac{(1.61-0.1341)^2 \times 246130 + (7.85-0.1341)^2 \times 4500 + ... + (558.00-0.1341)^2 \times 45}{3{,}780{,}000}$ and find the square root of the answer to obtain $\mathrm{sd}(X) = 2.2509$.

(d) The expected redeemable value in cents is calculated by (since we only incur a postage cost if we get a voucher)
E(X) − (*Postage Cost*) × pr(*getting a voucher*)
$= 13.41 - 40 \times 0.0677 = 10.74 \approx 11$ cents. Thus the expected cost of a box of Almond Delight is \$1.84 − \$0.11 = \$1.73. At \$1.60, the alternative brand is better value.

15. Let X = number of defective rivets in the sample.

(a) $X \sim$ Binomial($n = 8, p = 0.01$). Then pr($X \ge 2$) = 1 − pr($X \le 1$) = 0.00269.

(b) $Y \sim$ Binomial($n = 8, p = 0.02$). Then pr($X \le 1$) = 0.9897.

17. **(a)** Let X = number of attempts made.
From the case study:

x	1	2	3	4
pr($X = x$)	0.1	0.09	0.081	0.729

(b) $\mathrm{E}(X) = \sum x\,\mathrm{pr}(x) = 1 \times 0.1 + 2 \times 0.09 + 3 \times 0.081 + 4 \times 0.729 = 3.439$.
$\mathrm{sd}(X) = \sqrt{\sum (x-\mu)^2 \mathrm{pr}(x)} = \sqrt{(1 - 3.439)^2 \times 0.1 + ... + (4 - 3.439)^2 \times 0.729}$
$= \sqrt{1.0263} = 1.0131$.

(c) E(*Cost*) = \$7,000 × E(X) = \$7,000 × 3.439 = \$24,073.

(d) pr(*Still childless*) $= (\frac{9}{10})^4 = 0.6561$. [Alternatively, use pr$(Y = 0)$ where $Y \sim$ Binomial$(n = 4, p = 0.1)$.]

(e) and **(f)** The frequencies are computed in the following table.

Group	Starting number	Yes 1st	No	Yes 2nd	No	Yes 3rd	No	Yes 4th
1	30,000	6,000	24,000	4,800	19,200	3,840	15,360	3,072
2	30,000	3,000	27,000	2,700	24,300	2,430	21,870	2,187
3	40,000	400	39,600	396	39,204	392	38,812	388
Total	100,000	9,400	90,600	7,896	82,704	6,662	76,042	5,647

Here "Yes" means the number who were successful, "No" means the number unsuccessful and "1st" means first attempt, and so on.

(g) Proportion who succeed on first attempt $= \frac{9,400}{100,000} = 0.094$.
Proportion who succeed on second attempt $= \frac{7,896}{90,600} = 0.0872$.
Proportion who succeed on third attempt $= \frac{6,662}{82,704} = 0.0806$.
Proportion who succeed on fourth attempt $= \frac{5,647}{76,042} = 0.0743$.

19. Let X = number of incorrect identifications out of 91 calves. Then $X \sim$ Binomial$(n = 91, p)$ and you can compute pr$(X \leq 1)$ for different p, namely:

p	0	0.02	0.04	0.06	0.08
pr$(X \leq 1)$	1	0.4545	0.1167	0.0244	0.0045

By calculating a couple more points between 0.04 and 0.06, and plotting a graph, you can estimate that the value of p giving pr$(X \leq 1) = 0.1$ is 0.042.

21. If X is the number of mutations, then $X \sim$ Binomial$(n = 20,000, p = \frac{1}{10,000})$.

(a) pr$(X = 0) = 0.1353$.

(b) pr$(X \geq 1) = 1 -$ pr$(X = 0) = 0.8647$.

(c) pr$(X \leq 3) = 0.8571$.

23. **(a)** $X \sim$ Binomial$(n = 10, p = \frac{1}{50})$. The Binomial distribution is appropriate since the person either wins or does not win a bottle, sampling is with replacement so that the draws are (supposedly) independent, and the probability of winning is the same for each draw, namely, $\frac{1}{50}$.

(b) **(i)** pr$(X = 0) = 0.81707$, pr$(X \leq 2) = 0.999136$. **(ii)** pr$(X \geq 3) = 1 -$ pr$(X \leq 2) = 0.000864$.

(c) Since the probability in (b)(ii) is so small, the fact that one person has won three bottles would lead us to suspect that the names were not properly stirred.

(d) By b(ii), pr(E_i) = pr$(X \geq 3) \approx 0.00086 \neq 0$. If the event E_i has occurred for at least 3 people, then the event E_j $(j \neq i)$ cannot occur for the other people (since there are only 10 bottles to be won), so pr$(E_j) = 0$. This establishes that the outcome of E_i depends on the occurrence or otherwise of the other outcomes. Hence the E_i 's are not independent.

(e) Let Y = number of people who win 3 or more bottles. Assuming independence of the E_i's and regarding E_i as a binomial trial you have $Y \sim$ Binomial$(n = 50, p = 0.000864)$. Hence pr$(Y \geq 1) = 1 -$ pr$(Y = 0) = 0.0423$.

(f) Although the last probability in (e) is somewhat larger (slightly over 4 chances in 100), our opinion in (c) is not changed.

Chapter 6 Continuous Random Variables

Exercises on Section 6.2.2

Note: Recall that for continuous distributions like the Normal, we do not have to worry whether interval endpoints are included or excluded.

2. We use $X \sim \text{Normal}(\mu = 266, \sigma = 16)$.

- **(a)** $\text{pr}(X < 252) = \text{pr}(X \leq 252) = 0.1908$.
- **(b)** $\text{pr}(260 < X \leq 280) = \text{pr}(X \leq 280) - \text{pr}(X \leq 260) = 0.4554$.
- **(c)** $\text{pr}(X > 280) = 1 - \text{pr}(X \leq 280) = 0.1908$.

3. We use $X \sim \text{Normal}(\mu = 100, \sigma = 15)$.

- **(a)** $\text{pr}(X < 80) = 0.09121$.
- **(b)** $\text{pr}(85 < X \leq 110) = \text{pr}(X \leq 85) - \text{pr}(X \leq 110) = 0.5889$.
- **(c)** $\text{pr}(X > 120) = 1 - \text{pr}(X \leq 120) = 0.09121$.

Exercises on Section 6.2.3

2. **(a)** 249.4. **(b)** The 98th percentile, which is 298.9.
(c) The 10th percentile, which is 245.5. **(d)** The 70th percentile, which is 274.4.

3. IQs are given to the nearest whole number. **(a)** 113.
(b) The 99th percentile, which is 135. **(c)** The 30th percentile, which is 92.

Exercises on Section 6.2.4

1. **(a)** From the 10th percentile to the 90th percentile, or $[154.75, 170.65]$.
(b) From the 5th percentile to the 95th percentile, or $[152.50, 172.90]$.

2. **(a)** From the 20th percentile to the 80th percentile, or $[23.85, 30.75]$.
(b) From the 10th percentile to the 90th percentile, or $[22.05, 32.55]$.

Exercises on Section 6.3.1

1. **(a)** 280 is $\frac{280-266}{16} = 0.875$ sd's above the mean.
(b) 250 is $\frac{250-266}{16} = -1$, i.e., 1 sd below the mean.
(c) 270 is $\frac{270-266}{16} = 0.25$ sd's above the mean.

2. **(a)** 80 is $\frac{80-100}{15} = -1.3333$, i.e., 1.3333 sd's below the mean.
(b) 110 is $\frac{110-100}{15} = 0.6667$ sd's above the mean.
(c) 90 is $\frac{90-100}{15} = -0.6667$, i.e., 0.6667 sd's below the mean.

Exercises on Section 6.3.2

1. **(a)** $z = 1.9600$, $[\mu - z\sigma, \mu + z\sigma] = [234.6, 297.4]$. $z = 1.2816$, $[\mu - z\sigma, \mu + z\sigma] = [245.5, 286.5]$.

(b) $z = 1.6449$, $\mu + z\sigma = 292.3$.
$z = 2.3263$, $\mu + z\sigma = 303.2$.

(c) $z = 1.6449$, $\mu - z\sigma = 239.7$.
$z = 2.3263$, $\mu - z\sigma = 228.8$.

2. **(a)** $z = 1.96$, $[\mu - z\sigma, \mu + z\sigma] = [71, 129]$.
$z = 1.2816$, $[\mu - z\sigma, \mu + z\sigma] = [81, 119]$.

(b) $z = 1.6449$, $\mu + z\sigma = 125$.
$z = 2.3263$, $\mu + z\sigma = 135$.

(c) $z = 0.6745$, $\mu - z\sigma = 90$.
$z = 1.2816$, $\mu - z\sigma = 82$.

Exercises on Section 6.3.3

1. **(a)** **(i)** $\frac{-5-3}{4} = -2$. **(ii)** $\frac{11-3}{4} = 2$. **(iii)** $\frac{5-3}{4} = 0.5$. **(iv)** $\frac{1.4-3}{4} = -0.4$.

(b) **(i)** 2 sd's below. **(ii)** 2 sd's above. **(iii)** 0.5 sd's above. **(iv)** 0.4 sd's below.

2. **(a)** $\text{pr}(Z \geq \frac{5-7}{6} = -0.33) = 1 - \text{pr}(Z \leq -0.33) = 1 - 0.371 = 0.629$.

(b) $\text{pr}(Z \leq \frac{9-7}{6} = 0.33) = 0.629$.

(c) $\text{pr}(-0.33 = \frac{5-7}{6} \leq Z \leq \frac{11-7}{6} = 0.67) = \text{pr}(Z \leq 0.67) - \text{pr}(Z \leq -0.33) = 0.749 - 0.371 = 0.378$.

(d) $\text{pr}(-0.67 = \frac{3-7}{6} \leq Z \leq \frac{6-7}{6} = -0.17) = \text{pr}(Z \leq -0.17) - \text{pr}(Z \leq -0.67) = 0.433 - 0.251 = 0.182$.

(d) $\text{pr}(Z \leq \frac{3-7}{6} = -0.67) = 0.251$.

3. **(a)** $\text{pr}(Z \leq \frac{-4+3}{2} = -0.5) = 0.309$.

(b) $\text{pr}(Z \geq \frac{3}{2} = 1.5) = 1 - \text{pr}(Z \leq 1.5 = 1 - 0.933 = 0.067$.

(b) $\text{pr}(0 = \frac{-3+3}{2} \leq Z \leq \frac{-1+3}{2} = 1) = \text{pr}(Z \leq 1) - \text{pr}(Z \leq 0) = 0.841 - 0.5 = 0.341$.

Exercises on Section 6.4.3

1. **(a)** $\text{E}(Y) = 2.5 + 1.5 = 4.0$, $\text{sd}(X) = \sqrt{3^2 + 2^2} = 3.61$.

(b) $\text{E}(Y) = 2.5 - 1.5 = 1.0$, $\text{sd}(X) = \sqrt{3^2 + 2^2} = 3.61$.

(c) $\text{E}(Y) = 2.5 + 5 - 4 = 3.5$, $\text{sd}(X) = \sqrt{3^2 + 5^2 + 3^2} = 6.56$.

(d) $\text{E}(Y) = 1.5 + 5 = 6.5$, $\text{sd}(X) = \sqrt{2^2 + 5^2} = 5.39$.

(e) $\text{E}(Y) = 1.5 - 5 = -3.5$, $\text{sd}(X) = \sqrt{2^2 + 5^2} = 5.39$.

(f) $\text{E}(Y) = 2.5 + 1.5 - 5 = -1.0$, $\text{sd}(X) = \sqrt{3^2 + 2^2 + 5^2} = 6.16$.

(g) $\mathrm{E}(Y) = 5 - 4 = 1.0$, $\mathrm{sd}(X) = \sqrt{5^2 + 3^2} = 5.83$.

(h) $\mathrm{E}(Y) = 5 - (-4) = 9.0$, $\mathrm{sd}(X) = \sqrt{5^2 + 3^2} = 5.83$.

(i) $\mathrm{E}(Y) = 2.5 + 5 - (-4) = 11.5$, $\mathrm{sd}(X) = \sqrt{3^2 + 5^2 + 3^2} = 6.56$.

(j)

Variable	Mean		Standard deviation	
$W_1 = 2X_1$	2×2.5	$= 5.0$	2×3	$= 6$
$W_2 = 3X_2$	3×1.5	$= 4.5$	3×2	$= 6$
$Y = W_1 + W_2$	$5.0 + 4.5$	$= 9.5$	$\sqrt{6^2 + 6^2}$	$= 8.49$

(k)

Variable	Mean		Standard deviation	
$W_1 = 5X_1$	5×2.5	$= 12.5$	5×3	$= 15$
$W_2 = 4X_2$	4×1.5	$= 6.0$	4×2	$= 8$
$Y = W_1 - W_2$	$12.5 - 6.0$	$= 6.5$	$\sqrt{15^2 + 8^2}$	$= 17$

(l)

Variable	Mean		Standard deviation	
$W_1 = 2X_1$	2×2.5	$= 5.0$	2×3	$= 6$
$W_2 = 3X_2$	3×1.5	$= 4.5$	3×2	$= 6$
$W_3 = 4X_3$	4×5	$= 20.0$	4×5	$= 20$
$Y = W_1 + W_2 + W_3$	$5.0 + 4.5 + 20.0$	$= 29.5$	$\sqrt{6^2 + 6^2 + 20^2}$	$= 21.73.$

2. Let D be the score for a depressed child, and N be the score for a child not depressed. We have $D \sim \mathrm{Normal}(\mu = 11.2, \sigma = 6.8)$ and $N \sim \mathrm{Normal}(\mu = 8.5, \sigma = 7.8)$.

(a) We require $\mathrm{pr}(D < N)$ or $\mathrm{pr}(N - D \geq 0)$. Now $X = N - D \sim \mathrm{Normal}(\mu = 8.5 - 11.2, \sigma = \sqrt{7.8^2 + 6.8^2})$, i.e., $X \sim \mathrm{Normal}(\mu = -2.7, \sigma = 10.3480)$. Thus, $\mathrm{pr}(X > 0) = 1 - \mathrm{pr}(X \leq 0) = 0.3971$.

(b) Some of them may be depressed or have a high CDI score.

Review Exercises 6

Note: All of the probabilities and "inverse" values required in these review exercises were obtained directly using a computer.

1. **(a)** **(i)** $\mathrm{pr}(X > 141) = 1 - \mathrm{pr}(X \leq 141) = 0.06681$.

(ii) $\mathrm{pr}(120 < X < 132) = \mathrm{pr}(X < 132) - \mathrm{pr}(X < 120) = 0.4515$.

(iii) $\mathrm{pr}(X < 118.5) = 0.2266$.

(b) We have $z = -0.6$ which tells us that 120 is 0.6 sd's below the mean.

(c) We have $z = 1.4$ which tells us that 140 is 1.4 sd's above the mean.

(d) We need the 85th percentile, i.e., a so that $0.85 = \mathrm{pr}(X \leq a)$. We find $a = 136.4$.

(e) We need the 90th percentile, i.e., b so that $0.9 = \mathrm{pr}(X \leq b)$, We find $b = 138.8$.

(f) We need the 1st percentile, i.e., c so that $0.01 = \mathrm{pr}(X \leq c)$, We find $c = 102.7$.

(g) From the 5th to the 95th percentile, or $[109.6, 142.4]$.

(h) From the 20th to the 80th percentile, or $[117.6, 134.4]$.

(i) Total $= T \sim \mathrm{Normal}(\mu = 20 \times 126, \sigma = \sqrt{20} \times 10)$, or $\mathrm{Normal}(2520, 44.72)$.
$\overline{X} = \frac{T}{20} \sim \mathrm{Normal}(\mu = 126, \sigma = \frac{10}{\sqrt{20}})$, or $\mathrm{Normal}(126, 2.24)$.

3. Let X be the distance reached. Then $X \sim \text{Normal}(\mu = 125, \sigma = 10)$.

(a) $\text{pr}(X \geq 120) = 1 - \text{pr}(X < 120) = 0.6915$.

(b) We need the 5th percentile, i.e., x so that $\text{pr}(X \leq x) = 0.05$. We find that $x = 108.55$, i.e., about 109 cm.

(c) The pilot has a reach which is 1.5 sd's above the mean reach,
namely $125 + 1.5 \times 10 = 140$ cm.

5. Here $X \sim \text{Normal}(\mu_X = 266, \sigma_X = 16)$.

(a) $\text{pr}(X > 349) = 1 - \text{pr}(X \leq 349) = 1.0657 \times 10^{-7}$.

(b) Looks like a wrong decision! Gestation periods are virtually never as long as 349 days.

(c) Let Y be the number of gestation periods lasting 349 or more days.
Then $Y \sim \text{Binomial}(n = 10{,}000{,}000, p = 1.0657 \times 10^{-7})$
and $\text{pr}(Y \geq 1) = 1 - \text{pr}(Y = 0) = 0.6555$. It reasonably likely that this would happen in at least one of 10 million births.
[We would put very little trust in these calculations. The weakest point is (a). It relies critically on a Normal approximation to the unknown true distribution of gestation periods working well in the far tail of the distribution; this approach is highly suspect.]

7. Let X_H be the serum acid of a randomly chosen healthy person.
Let X_G be the serum acid of a randomly chosen gout sufferer.
Then $X_H \sim \text{Normal}(\mu = 5.0, \sigma = 1)$, and $X_G \sim \text{Normal}(\mu = 8.5, \sigma = 1)$.

(a) $\text{pr}(X_G < 6.75) = 0.0401$.

(b) $\text{pr}(X_H > 6.75) = 0.0401$.

(c) $X \sim \text{Binomial}(n = 50, p = 0.0401)$.

(d) Let u be the new cutoff level for serum uric acid. Then $\text{pr}(X_G > u) = 0.90$ and $u = 7.2184$.

(e) $\text{pr}(X_H > 7.2184) = 0.0133$.

9. **(a)** We use the $\text{Normal}(\mu = 106, \sigma = 5)$ distribution. Then, $\text{pr}(X \leq 100) = 0.1151$, or 11.5%.

(b) $Y \sim \text{Binomial}(n = 100, p = 0.1151)$. [The value of p is our answer from (a).]

(c) Here Y is the number sold at discount giving a profit of $(65 - 25)Y$. The remainding $100 - Y$ are sold at full price giving a profit of $(70 - 26.5) \times (100 - Y)$.
The total profit is $P = (65 - 25)Y + (70 - 26.5)(100 - Y) = 4350 - 3.5Y$.
$\text{E}(P) = 4350 - 3.5\text{E}(Y) = 4350 - 3.5 \times 11.51 = 4310$, or \$43.10.

***(d)** $\text{pr}(X < 95 \,|\, X < 100) = \frac{\text{pr}(X<95 \text{ and } X<100)}{\text{pr}(X<100)} = \frac{\text{pr}(X<95)}{\text{pr}(X<100)} = \frac{0.01390}{0.1151} = 0.1208$, or 12.1%.

11. Let F = length of randomly chosen female and M = the length of a randomly chosen male. Assume $F \sim \text{Normal}(\mu_F = 89.2, \sigma_F = 6.6)$ and $M \sim \text{Normal}(\mu_M = 92.0, \sigma_M = 6.7)$. Using these two distributions we find the following.

(a) We need the 90th percentile of the female distribution, i.e., we need a so that $\text{pr}(F \le a) = 0.9$. We find that $a = 97.66\,\text{cm}$. Moreover, using the male distribution, we find that $\text{pr}(M \ge 97.66) = 1 - \text{pr}(M < 97.66) = 0.1992$.

(b) $\text{pr}(F > M) = \text{pr}(Y > 0)$ where $Y = F - M$. Now, $Y \sim \text{Normal}(\mu_Y = 89.2 - 92.0 = -2.8, \sigma_Y = \sqrt{6.6^2 + 6.7^2} = 9.4048)$ and $\text{pr}(Y \ge 0) = 1 - \text{pr}(Y < 0) = 0.3830$.

(c) $\text{pr}(M - F \ge 10) = \text{pr}(Y \le -10) = 0.2220$.

(d) Human couples tend to be related to some extent in their physical dimensions rather than independent. For example, very tall women seldom partner with very short men. Perhaps similar dynamics affect coyotes.

13. (a) Let X be the weight of an adult. Then $X \sim \text{Normal}(\mu = 73, \sigma = 13)$. Then the distribution of the sum of the weights of 11 randomly sampled people is Normally distributed with mean $n\,\mu = 11 \times 73$ and standard deviation $\sqrt{N}\,\sigma = \sqrt{11} \times 13)$, i.e., $\mathit{Sum} \sim \text{Normal}(803, 43.1161)$. Then $\text{pr}(\mathit{Sum} > 800) = 1 - \text{pr}(\mathit{Sum} \le 800) = 0.5277$. We have assumed that the weights of adults satisfy a single Normal distribution, and that the weights are independent. Both assumptions are doubtful because of sex differences in weight, and friends or acquaintances often take a lift together.

Let F the weight of a random female, and M the weight of a random male. Then, $F \sim \text{Normal}(\mu_F = 68, \sigma_F = 12)$ and $M \sim \text{Normal}(\mu_M = 78, \sigma_M = 12)$.

(b) The distribution of the sum of the weights of 11 randomly sampled men, B, is Normal with $\mu_B = 11 \times 78 = 858$, and $\sigma_B = \sqrt{11} \times 12 = 39.7995$. Then $\text{pr}(B > 800) = 1 - \text{pr}(B \le 800) = 0.9275$.

(c)

Variable	Mean		Standard deviation
S_{7M} = sum of wts of 7 men	7×78	$= 546$	$\sqrt{7} \times 12 = 31.74902$
S_{4W} = sum of wts of 4 women	4×68	$= 272$	$\sqrt{4} \times 12 = 24$
$C = S_{7M} + S_{4W}$	$546 + 272$	$= 818$	$\sqrt{31.74902^2 + 24^2}$ $= 39.7995$

Since $C \sim \text{Normal}(818, 39.7995)$, $\text{pr}(C > 800) = 1 - \text{pr}(C \le 800) = 0.6745$.

(d) We need $\text{pr}(C > B) = \text{pr}(D > 0)$, where $D = C - B$. The distribution of D is Normal with mean $\mu_D = \mu_C - \mu_B = 818 - 858 = -40$ and standard deviation $\sigma_D = \sqrt{39.7995^2 + 39.7995^2} = 56.2857$. Then, $\text{pr}(D > 0) = 0.2386$.

***(e)** Assume they are all men and there are m of them with total weight T.
Then, $T \sim \text{Normal}(m \times 78, \sqrt{m} \times 12)$. Using this distribution, we calculated the value of $\text{pr}(T > 800)$ for each of several values of m.

m	7	8	9	10	11
$\text{pr}(T > 800)$	6.7×10^{-16}	1.1×10^{-07}	0.0032	0.2991	0.9275

If we choose $m = 9$, there is still only a very small chance that 9 men will overload the elevator, whereas there is almost a 30% chance of an overload with 10 men. We recommend a limit of 9.

(f) University students will be lighter on average, as people tend to gain weight with age.

15. Since the distribution of the weight of a single random ingot is Normal($\mu_I = 500, \sigma_I = 10$), the total weight of for $n = 100$ ingots is Normal with mean $100 \times 500\,\text{g} = 50\,\text{kg}$ and standard deviation $\sqrt{100} \times 10\,\text{g} = 0.1\,\text{kg}$, i.e., $Sum \sim$ Normal(50 kg, 0.1 kg).

(a) pr($49.9 < Sum < 50.1$) = pr($Sum < 50.1$) − pr($Sum \leq 49.9$) = 0.6827.

***(b)** Let X be the number of weighings. Then pr($X = 1$) = pr($Sum \geq 49.9$) = 0.8413, and pr($X = 101$) = pr($Sum < 49.9$) = 0.1587.
Hence $\mathrm{E}(X) = \sum x\,\mathrm{pr}(x) = 1 \times 0.8413 + 101 \times 0.1587 = 16.87$, or about 17.

17. **(a)** **(i)** Since X is a sum from a sample of size $n = 50$ sheets, $X \sim$ Normal($\mu = 50 \times .5, \sigma = \sqrt{50} \times .05$), i.e., Normal(25, 0.35355).

(ii) pr($X > 26$) = 0.00234.

(b) $Y \sim$ Normal($\mu = 49 \times 0.05, \sigma = 7 \times 0.02$), i.e., Normal($\mu = 2.45, \sigma = 0.14$).

(c) $W = X + Y \sim$ Normal($\mu = 25 + 2.45 = 27.45, \sigma = \sqrt{0.35355^2 + 0.14^2} = 0.3803$) and pr($W > 28.3$) = 0.0127.

(d) **(i)** We no longer have a sum made up of independently varying thicknesses, but now have 49 copies of one random thickness, so $M \sim$ Normal($\mu = 49 \times .05, \sigma = 49 \times .02$), i.e., Normal(2.45, 0.98).

(ii) $X + M \sim$ Normal($\mu = 27.45, \sqrt{0.35355^2 + 0.98^2} = 1.0418$) and pr($X + M > 28.3$) = 0.2073

(e) We have essentially answered this in (d)(i). Y is the sum of 49 independently varying thicknesses, whereas M is made up of 49 copies of the same random thickness. Thus, M is much more variable than Y (see page 265 of the book).

19. **(a)** Let C be the weight of a randomly chosen carton. Then $C \sim$ Normal($\mu_C = 100, \sigma_C = 1.25$). A carton is strapped with probability pr($C > 101$) = 0.2119.
If there are 64 cartons, then the number to be strapped is Binomial($n = 64, p = 0.2119$).

(b) Let W be the weight of a pallet. Then $W \sim$ Normal($\mu_W = 150, \sigma_W = 3$).

Let S_C be the total weight (a sum) for 64 independent cartons. Then the distribution of S_C is Normal with mean $\mu_S = 64 \times 100 = 6400$ and $\sigma_S = \sqrt{64} \times 1.25 = 10$.

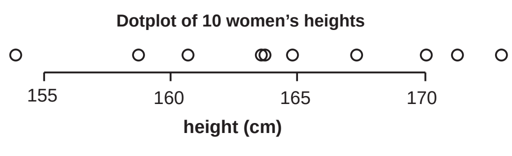
Let T be the total weight of a loaded pallet, then the distribution of $T = W + S_C$ is Normal with $\mu_T = 150 + 6400 = 6550$, and standard deviation $\sigma_T = \sqrt{3^2 + 10^2} = 10.4403$.

We require the 5th percentile of the distribution of T, i.e., t such that pr($T < t$) = 0.05. We find that $t = 6533$ kg.

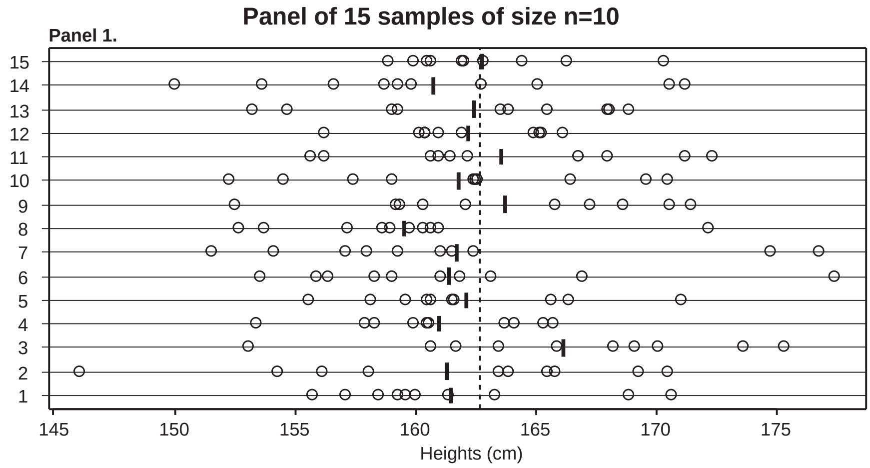
21. **(a)** The "heights of ten women" randomly sampled from a Normal distribution with $\mu = 162.7$ cm and $\sigma = 6.2$ cm are given below.

158.7860	171.3006	164.8145	170.0762	163.7403
173.0414	153.9394	163.6225	160.6812	167.3418

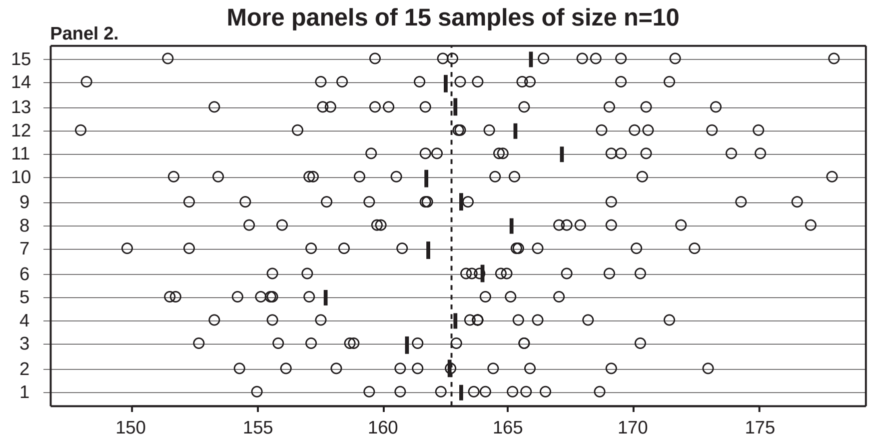
The corresponding dot plot of the 10 observations follows.



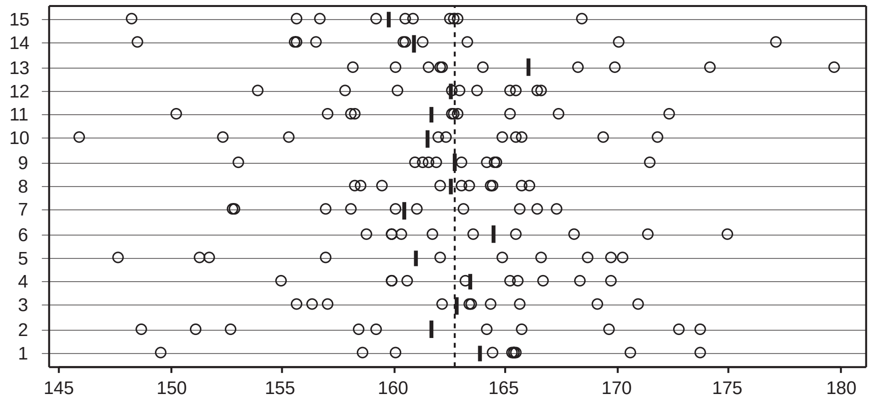
(b) A dot plot of fifteen samples each of ten women is given below. The dotted vertical line represents the population mean of 162.7cm and the sample means for each of the fifteen samples are represented by small vertical bars. A further three sets of 15 follow. Each line in a plot corresponds to a single sample.



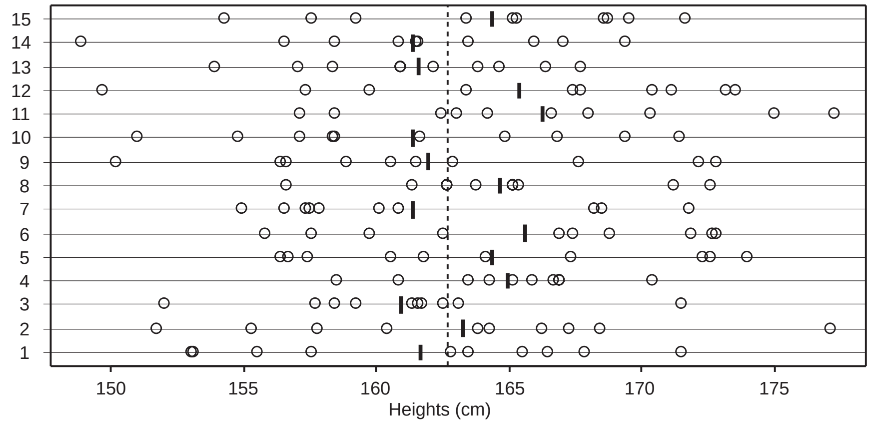
Differences in center are very easy to see as the centers are marked by the vertical bars and they vary greatly. For other features, we have picked out some of the more extreme examples. For extreme differences in spread, compare sample 1 and sample 14 or 15 in panel 2, or sample 8 in panel 3 with almost any other sample. There are many cases of extreme outliers, e.g. see on the right of sample 8 of panel 1. Sample 5 in panel 2 is one of the more extreme examples of a gap in the middle of a data set. Sample 13 in panel 3 looks rather right skewed while sample 12 looks quite left skewed. Remember what we are seeing is “perfect data” from a Normal random number generator. There are no mistakes here.



Panel 3.



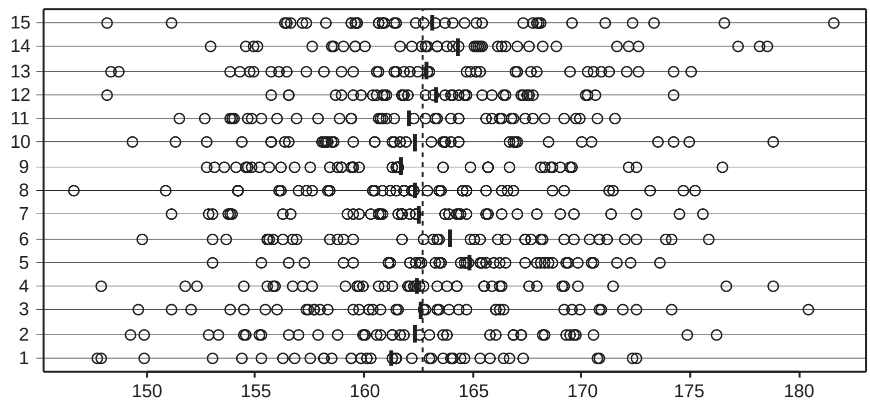
Panel 4.

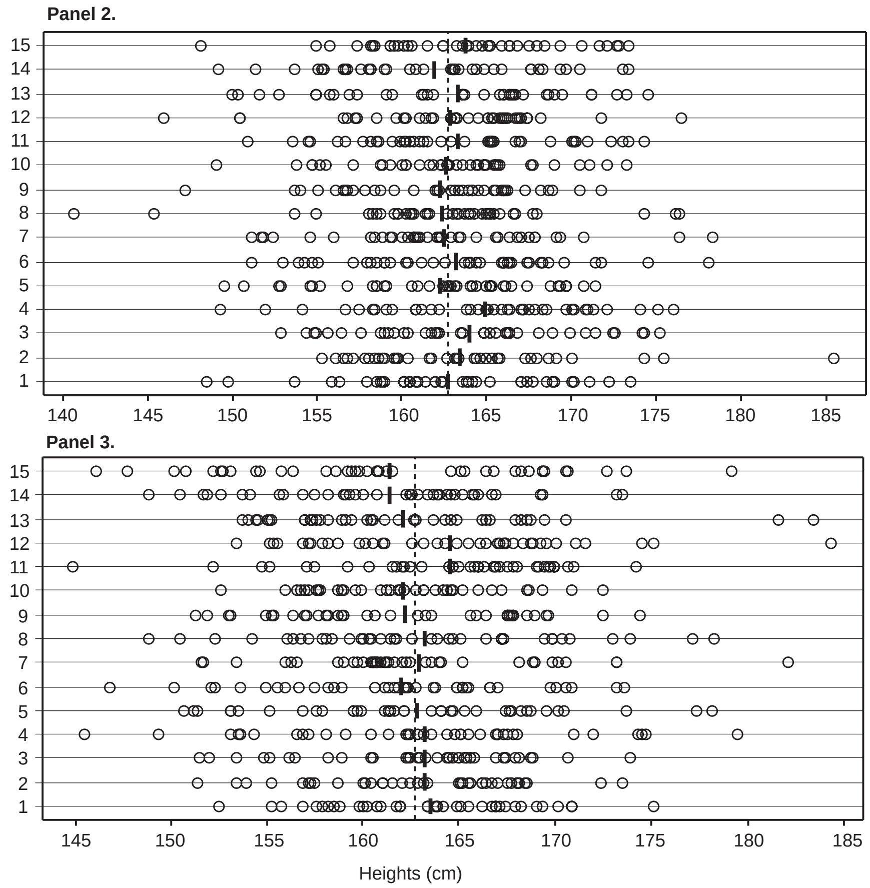


(c) Dot plots for three sets of fifteen samples each of 40 women follow. The variation in means is clearly much smaller. The "outliers" are seldom as extreme, there is less variation in spreads from sample to sample, and the gaps tend to be narrower.

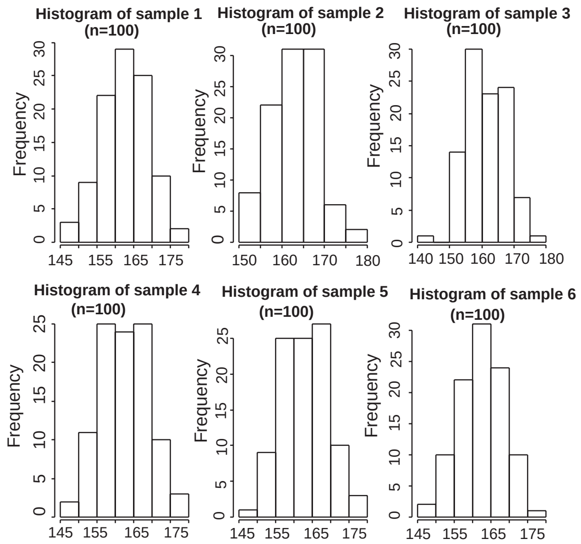
Panels of 15 samples of size n=40

Panel 1.





(d) Histograms for 6 samples each of size 100 is given below. Athough the bulk of the observations are always falling within very much the same range and all are unimodal, at a more detailed level the shapes of the histograms vary considerably. The two top right ones even suggest skewness.



(e) Histograms for 6 samples each of size 1000 is given below. When we have batches of 1000 observations the histograms are more bell shaped and there is a little less variation in histogram shape from sample to sample, but the variation is still noticeable. We find later that if we take enormous samples, e.g. millions, we get reliably bell-shaped histograms that vary very little.