Supplement for "Local regression for vector responses" by A. H. Welsh and T. W. Yee (2006). *Journal of Statistical Planning and Inference*, 136(9): 3007–3031.

Appendix A. Proof of Theorem 2

Case A

For the diagonal block matrices of $\mathbf{\Delta}_n(a)$, we can apply standard arguments to show that, for $t > 0$,

$$
\begin{aligned}
n^{-1} \sum_{i=1}^{n} \alpha_i \, K_{h_j}(x_i^{(j)} - x)^t \;&\sim\; \overline{\alpha}\, h_j^{1-t}\, g_j(x) \int K_j(z)^t \, dz, \\
n^{-1} \sum_{i=1}^{n} \alpha_i \, (x_i^{(j)} - x)\, K_{h_j}(x_i^{(j)} - x)^t \;&\sim\; \overline{\alpha}\, h_j^{3-t}\, g_j'(x) \int z^2 K_j(z)^t \, dz, \\
n^{-1} \sum_{i=1}^{n} \alpha_i \, (x_i^{(j)} - x)^2\, K_{h_j}(x_i^{(j)} - x)^t \;&\sim\; \overline{\alpha}\, h_j^{3-t}\, g_j(x) \int z^2 K_j(z)^t \, dz,
\end{aligned}
$$

and for the off-diagonal block matrices of $\mathbf{\Delta}_n(a)$,

$$
n^{-1} \sum_{i=1}^{n} \beta_i \, K_{h_1}(x_i^{(1)} - x)^a \, K_{h_2}(x_i^{(2)} - x)^{1-a} \;\sim
$$

$$
\begin{cases}
\overline{\beta}\, h_1^{1-a}\, h_2^{a}\, g(\mathbf{x}) \int K_1(z)^a \, dz \int K_2(z)^{1-a} \, dz, & 0 < a < 1, \\
\overline{\beta}\, g_1(x), & a = 1, \\
\overline{\beta}\, g_2(x), & a = 0;
\end{cases}
$$

$$
n^{-1} \sum_{i=1}^{n} \beta_i \, K_{h_1}(x_i^{(1)} - x)^a \, K_{h_2}(x_i^{(2)} - x)^{1-a} (x_i^{(1)} - x) \;\sim
$$

$$
\begin{cases}
\overline{\beta}\, h_1^{3-a}\, h_2^{a}\, g^{(1)}(\mathbf{x}) \int z^2 K_1(z)^a \, dz \int K_2(z)^{1-a} \, dz, & 0 < a < 1, \\
\overline{\beta}\, h_1^2 g_1'(x) \mu_2^{(1)}, & a = 1, \\
\overline{\beta}\, g_2(x) \{E(x_i^{(1)} | x_i^{(2)} = x) - x\}, & a = 0;
\end{cases}
$$

$$
n^{-1} \sum_{i=1}^{n} \beta_i \, K_{h_1}(x_i^{(1)} - x)^a \, K_{h_2}(x_i^{(2)} - x)^{1-a} (x_i^{(2)} - x) \;\sim
$$

$$
\begin{cases}
\overline{\beta}\, h_1^{1-a}\, h_2^{2+a}\, g^{(2)}(\mathbf{x}) \int K_1(z)^a \, dz \int z^2 K_2(z)^{1-a} \, dz, & 0 < a < 1, \\
\overline{\beta}\, g_1(x) \{E(x_i^{(2)} | x_i^{(1)} = x) - x\}, & a = 1, \\
\overline{\beta}\, h_2^2 g_2'(x) \mu_2^{(2)}, & a = 0;
\end{cases}
$$

$$n^{-1}\sum_{i=1}^{n}\beta_i\,K_{h_1}(x_i^{(1)}-x)^a\,K_{h_2}(x_i^{(2)}-x)^{1-a}(x_i^{(1)}-x)\,(x_i^{(2)}-x)\ \sim$$

$$\begin{cases}\overline{\beta}\,h_1^{3-a}\,h_2^{2+a}\,g^{(1,2)}(\mathbf{x})\,\int z^2K_1(z)^a\,dz\,\int z^2K_2(z)^{1-a}\,dz, & 0<a<1,\\ \overline{\beta}\,h_1^2\mu_2^{(1)}m_1^{(2)}(\mathbf{x}), & a=1,\\ \overline{\beta}\,h_2^2\mu_2^{(2)}m_1^{(1)}(\mathbf{x}), & a=0.\end{cases}$$

For $0<a<1$, it follows that the limit of $\mathbf{\Delta}_n(a)$ is

$$\mathbf{\Delta}(a)\ =\ \begin{pmatrix}\overline{\alpha}\,\mathbf{\Delta}_{11} & \overline{\beta}\,\mathbf{\Delta}_{12}(a)\\ \overline{\beta}\,\mathbf{\Delta}_{21}(a) & \overline{\gamma}\,\mathbf{\Delta}_{22}\end{pmatrix},$$

where

$$\mathbf{\Delta}_{jj}\ =\ \begin{pmatrix}g_j(x) & h_j^2\,g_j'(x)\,\mu_2^{(j)}\\ h_j^2\,g_j'(x)\,\mu_2^{(j)} & h_j^2\,g_j(x)\,\mu_2^{(j)}\end{pmatrix},$$

$$\mathbf{\Delta}_{12}(a)\ =\ \begin{pmatrix}h_1^{1-a}\,h_2^a\,g(\mathbf{x})\int K_1(z)^a\,dz\,\int K_2(z)^{1-a}\,dz & h_1^{1-a}\,h_2^{2+a}\,g^{(2)}(\mathbf{x})\int K_1(z)^a\,dz\,\int z^2\,K_2(z)^{1-a}\,dz\\ h_1^{3-a}\,h_2^a\,g^{(1)}(\mathbf{x})\int z^2\,K_1(z)^a\,dz\,\int K_2(z)^{1-a}\,dz & h_1^{3-a}\,h_2^{2+a}\,g^{(1,2)}(\mathbf{x})\int z^2\,K_1(z)^a\,dz\,\int z^2\,K_2(z)^{1-a}\,dz\end{pmatrix},$$

and

$$\mathbf{\Delta}_{21}(a)\ =\ \begin{pmatrix}h_1^a\,h_2^{1-a}\,g(\mathbf{x})\int K_1(z)^{1-a}\,dz\,\int K_2(z)^a\,dz & h_1^{2+a}\,h_2^{1-a}\,g^{(1)}(\mathbf{x})\int z^2\,K_1(z)^{1-a}\,dz\,\int K_2(z)^a\,dz\\ h_1^a\,h_2^{3-a}\,g^{(2)}(\mathbf{x})\int K_1(z)^{1-a}\,dz\,\int z^2\,K_2(z)^a\,dz & h_1^{2+a}\,h_2^{3-a}\,g^{(1,2)}(\mathbf{x})\int z^2\,K_1(z)^{1-a}\,dz\,\int z^2\,K_2(z)^a\,dz\end{pmatrix}.$$

Using the fact that $h_1/h_2\longrightarrow 1$, we can write $h_1\sim h$, $h_2\sim h$ and

$$\mathbf{\Delta}(a)\ \sim\ \begin{pmatrix}\overline{\alpha}\,d_{11} & h^2\,\overline{\alpha}\,d_{12} & h\,\overline{\beta}\,d_{13}(a) & h^3\,\overline{\beta}\,d_{14}(a)\\ h^2\,\overline{\alpha}\,d_{12} & h^2\,\overline{\alpha}\,d_{22} & h^3\,\overline{\beta}\,d_{23}(a) & h^5\,\overline{\beta}\,d_{24}(a)\\ h\,\overline{\beta}\,d_{31}(a) & h^3\,\overline{\beta}\,d_{32}(a) & \overline{\gamma}\,d_{33} & h^2\overline{\gamma}\,d_{34}\\ h^3\,\overline{\beta}\,d_{41}(a) & h^5\,\overline{\beta}\,d_{42}(a) & h^2\,\overline{\gamma}\,d_{34} & h^2\,\overline{\gamma}\,d_{44}\end{pmatrix}.$$

Then, we can show that

$$\mathbf{\Delta}^{-1}(a)\ \sim\ \begin{pmatrix}\overline{\alpha}^{-1}\mathbf{\Delta}_{11}^{-1} & h\frac{\overline{\beta}}{\overline{\alpha}\,\overline{\gamma}}\mathbf{\Delta}^{12}(a)\\ h\frac{\overline{\beta}}{\overline{\alpha}\,\overline{\gamma}}\mathbf{\Delta}^{21}(a) & \overline{\gamma}^{-1}\mathbf{\Delta}_{22}^{-1}\end{pmatrix}, \tag{1}$$

where

$$\boldsymbol{\Delta}_{jj}^{-1}(a) \;\sim\; \frac{1}{g_j(x)} \begin{pmatrix} 1 & -g_j'(x)/g_j(x) \\ -g_j'(x)/g_j(x) & 1/(h_j^2\,\mu_2^{(j)}) \end{pmatrix}.$$

As the off diagonal blocks are of smaller order than the diagonal blocks, they do not contribute to the limits so we do not need to give them explicitly.

Next, for $\mathbf{t}_n(a)$, we have

$$n^{-1}\sum_{i=1}^{n} \alpha_i\, K_{h_1}(x_i^{(1)}-x)\,(x_i^{(1)}-x)^3 \;\sim\; \overline{\alpha}\, h_1^4\, g_1'(x)\, \mu_4^{(1)},$$

$$n^{-1}\sum_{i=1}^{n} \alpha_i\, K_{h_1}(x_i^{(1)}-x)\,(x_i^{(1)}-x)^4 \;\sim\; \overline{\alpha}\, h_1^4\, g_1(x)\, \mu_4^{(1)},$$

$$n^{-1}\sum_{i=1}^{n} \beta_i\, K_{h_1}(x_i^{(1)}-x)^a\, K_{h_2}(x_i^{(2)}-x)^{1-a}\,(x_i^{(2)}-x)^2 \;\sim$$

$$\begin{cases} \overline{\beta}\, h_1^{1-a}\, h_2^{2+a}\, g(\mathbf{x}) \int K_1(z)^a\, dz \int z^2\, K_2(z)^{1-a}\, dz, & 0<a<1, \\ \overline{\beta}\, g_1(x)\, E\left[\left(x_i^{(2)}-x\right)^2 \middle| x_i^{(1)}=x\right], & a=1, \\ \overline{\beta}\, h_2^2\, g_2(x)\, \mu_2^{(2)}, & a=0; \end{cases}$$

$$n^{-1}\sum_{i=1}^{n} \beta_i\, K_{h_1}(x_i^{(1)}-x)^a\, K_{h_2}(x_i^{(2)}-x)^{1-a}\,(x_i^{(2)}-x)^3 \;\sim$$

$$\begin{cases} \overline{\beta}\, h_1^{1-a}\, h_2^{4+a}\, g^{(2)}(\mathbf{x}) \int K_1(z)^a\, dz \int z^4\, K_2(z)^{1-a}\, dz, & 0<a<1, \\ \overline{\beta}\, g_1(x)\, E\left[\left(x_i^{(2)}-x\right)^3 \middle| x_i^{(1)}=x\right], & a=1, \\ \overline{\beta}\, h_2^4\, g_2'(x)\, \mu_4^{(2)}, & a=0; \end{cases}$$

$$n^{-1}\sum_{i=1}^{n} \beta_i\, K_{h_1}(x_i^{(1)}-x)^a\, K_{h_2}(x_i^{(2)}-x)^{1-a}\,(x_i^{(1)}-x)\,(x_i^{(2)}-x)^2 \;\sim$$

$$\begin{cases} \overline{\beta}\, h_1^{3-a}\, h_2^{2+a}\, g^{(1)}(\mathbf{x}) \int z^2\, K_1(z)^a\, dz \int z^2\, K_2(z)^{1-a}\, dz, & 0<a<1, \\ \overline{\beta}\, h_1^2\, \mu_2^{(1)}\, m_2^{(2)}(\mathbf{x}), & a=1, \\ \overline{\beta}\, h_2^2\, \mu_2^{(2)}\, g_2(x)\, \left\{E\left[x_i^{(1)} \middle| x_i^{(2)}=x\right]-x\right\}, & a=0; \end{cases}$$

and

$$n^{-1}\sum_{i=1}^{n} \beta_i\, K_{h_1}(x_i^{(1)}-x)^a\, K_{h_2}(x_i^{(2)}-x)^{1-a}\,(x_i^{(1)}-x)\,(x_i^{(2)}-x)^3 \;\sim$$

$$\begin{cases} \overline{\beta}\, h_1^{3-a}\, h_2^{4+a}\, g^{(1,2)}(\mathbf{x}) \int z^2\, K_1(z)^a\, dz \int z^4\, K_2(z)^{1-a}\, dz, & 0<a<1, \\ \overline{\beta}\, h_1^2\, \mu_2^{(1)}\, m_2^{(2)}(\mathbf{x}), & a=1, \\ \overline{\beta}\, h_2^4\, \mu_4^{(2)}\, m_1^{(1)}(\mathbf{x}), & a=0. \end{cases}$$

For $0 < a < 1$, substituting these results into (16) gives

$$\mathbf{t}(a) \;\sim\; \begin{pmatrix} h_1^2\,\overline{\alpha}\,g_1(x)\,\mu_2^{(1)}\,f_1''(x)/2 \\ \\ h_1^4\,\overline{\alpha}\,\mu_4^{(1)}\,\{g_1'(x)\,f_1''(x)/2 + g_1(x)\,f_1'''(x)/3!\} \\ \\ h_2^2\,\overline{\gamma}\,g_2(x)\,\mu_2^{(2)}\,f_2''(x)/2 \\ \\ h_2^4\,\overline{\gamma}\,\mu_4^{(2)}\,\{g_2'(x)\,f_2''(x)/2 + g_2(x)\,f_2'''(x)/3!\} \end{pmatrix}. \tag{2}$$

Writing $\mathbf{t}(a) \sim (h_1^2\overline{\alpha}t_1, h_1^4\overline{\alpha}t_2, h_2^2\overline{\gamma}t_3, h_2^4\overline{\gamma}t_4)^T$, we can show that

$$\boldsymbol{\Delta}^{-1}(a)\,\mathbf{t}(a) \;\sim\; \begin{pmatrix} h_1^2 t_1\delta_{11} \\ h_1^2(t_1\delta_{12} + t_2\delta_{22}) \\ h_2^2 t_3\delta_{33} \\ h_2^2(t_3\delta_{34} + t_4\delta_{44}) \end{pmatrix} \;=\; \begin{pmatrix} h_1^2 t_1/d_{11} \\ h_1^2(t_2 - t_1 d_{12}/d_{11})/d_{22} \\ h_2^2 t_3/d_{33} \\ h_2^2(t_4 - t_3 d_{34}/d_{33})/d_{44} \end{pmatrix}. \tag{3}$$

Substituting from (1) and (2) for the terms on the right hand side of (3) gives the required expression for the bias.

Similar arguments using the appropriate forms for $\boldsymbol{\Delta}(a)$ and $\mathbf{t}(a)$ yield the asymptotic bias for $a = 0$ and $a = 1$.

Now consider $\boldsymbol{\theta}_n(a)$. For $0 < a < 1$, it is straightforward to show that

$$\begin{aligned} (\boldsymbol{\Theta}_{n11}(a))_{11} &\sim \frac{1}{nh_1}\left(n^{-1}\sum_{i=1}^{n}\alpha_i^2\sigma_{1i}^2\right) g_1(x)\,\nu_0^{(1)}, \\ (\boldsymbol{\Theta}_{n11}(a))_{12} &\sim \frac{h_1}{n}\left(n^{-1}\sum_{i=1}^{n}\alpha_i^2\sigma_{1i}^2\right) g_1'(x)\,\nu_2^{(1)}, \\ (\boldsymbol{\Theta}_{n11}(a))_{22} &\sim \frac{h_1}{n}\left(n^{-1}\sum_{i=1}^{n}\alpha_i^2\sigma_{1i}^2\right) g_1(x)\,\nu_2^{(1)}, \end{aligned}$$

$$\begin{aligned} (\boldsymbol{\Theta}_{n12}(a))_{11} \;&\sim\; \frac{1}{n}\left(n^{-1}\sum_{i=1}^{n}\alpha_i\beta_i\sigma_{1i}^2\right)(h_2/h_1)^{1-a}\;g(\mathbf{x})\int K_1(z)^{2-a}\,dz\;\int K_2(z)^a\,dz \\ &+\; \frac{1}{n}\left(n^{-1}\sum_{i=1}^{n}(\alpha_i\gamma_i + \beta_i^2)\sigma_{1i}\sigma_{2i}\rho_i\right) g(\mathbf{x}) \\ &+\; \frac{1}{n}\left(n^{-1}\sum_{i=1}^{n}\gamma_i\beta_i\sigma_{2i}^2\right)(h_1/h_2)^{1-a}\;g(\mathbf{x})\int K_1(z)^a\,dz\;\int K_2(z)^{2-a}\,dz, \end{aligned}$$

$$
\begin{aligned}
(\boldsymbol{\Theta}_{n12}(a))_{21} \;\sim\;& \frac{h_1h_2}{n}\left(n^{-1}\sum_{i=1}^{n}\alpha_i\beta_i\sigma_{1i}^2\right)(h_1/h_2)^a\, g^{(1)}(\mathbf{x})\int z^2K_1(z)^{2-a}\,dz\;\int K_2(z)^a\,dz\\
&+\;\frac{h_1^2}{n}\left(n^{-1}\sum_{i=1}^{n}(\alpha_i\gamma_i+\beta_i^2)\sigma_{1i}\sigma_{2i}\rho_i\right)\; g^{(1)}(\mathbf{x})\,\mu_2^{(1)}\\
&+\;\frac{h_1^2}{n}\left(n^{-1}\sum_{i=1}^{n}\gamma_i\beta_i\sigma_{2i}^2\right)(h_1/h_2)^{1-a}\, g^{(1)}(\mathbf{x})\int z^2K_1(z)^{a}\,dz\;\int K_2(z)^{2-a}\,dz,\\
(\boldsymbol{\Theta}_{n12}(a))_{12} \;\sim\;& \frac{h_2^2}{n}\left(n^{-1}\sum_{i=1}^{n}\alpha_i\beta_i\sigma_{1i}^2\right)(h_2/h_1)^{1-a}\, g^{(2)}(\mathbf{x})\int K_1(z)^{2-a}\,dz\;\int z^2K_2(z)^a\,dz\\
&+\;\frac{h_2^2}{n}\left(n^{-1}\sum_{i=1}^{n}(\alpha_i\gamma_i+\beta_i^2)\sigma_{1i}\sigma_{2i}\rho_i\right)\; g^{(2)}(\mathbf{x})\,\mu_2^{(2)}\\
&+\;\frac{h_1h_2}{n}\left(n^{-1}\sum_{i=1}^{n}\gamma_i\beta_i\sigma_{2i}^2\right)(h_2/h_1)^{a}\, g^{(2)}(\mathbf{x})\int K_1(z)^{a}\,dz\;\int z^2K_2(z)^{2-a}\,dz,\\
(\boldsymbol{\Theta}_{n12}(a))_{22} \;\sim\;& \frac{h_1h_2^3}{n}\left(n^{-1}\sum_{i=1}^{n}\alpha_i\beta_i\sigma_{1i}^2\right)(h_1/h_2)^a\, g^{(1,2)}(\mathbf{x})\int z^2K_1(z)^{2-a}\,dz\;\int z^2K_2(z)^a\,dz\\
&+\;\frac{h_1^2h_2^2}{n}\left(n^{-1}\sum_{i=1}^{n}(\alpha_i\gamma_i+\beta_i^2)\sigma_{1i}\sigma_{2i}\rho_i\right)\; g^{(1,2)}(\mathbf{x})\,\mu_2^{(1)}\mu_2^{(2)}\\
&+\;\frac{h_1^3h_2}{n}\left(n^{-1}\sum_{i=1}^{n}\gamma_i\beta_i\sigma_{2i}^2\right)(h_2/h_1)^{a}\, g^{(1,2)}(\mathbf{x})\int z^2K_1(z)^{a}\,dz\;\int z^2K_2(z)^{2-a}\,dz,
\end{aligned}
$$

and

$$
\begin{aligned}
(\boldsymbol{\Theta}_{n22}(a))_{11} \;\sim\;& \frac{1}{n\,h_2}\left(n^{-1}\sum_{i=1}^{n}\gamma_i^2\sigma_{2i}^2\right) g_2(x)\,\nu_0^{(2)},\\
(\boldsymbol{\Theta}_{n22}(a))_{12} \;\sim\;& \frac{h_2}{n}\left(n^{-1}\sum_{i=1}^{n}\gamma_i^2\sigma_{2i}^2\right) g_2'(x)\,\nu_2^{(2)},\\
(\boldsymbol{\Theta}_{n22}(a))_{22} \;\sim\;& \frac{h_2}{n}\left(n^{-1}\sum_{i=1}^{n}\gamma_i^2\sigma_{2i}^2\right) g_2(x)\,\nu_2^{(2)}.
\end{aligned}
$$

We can write

$$
\boldsymbol{\Theta}_n(a) \sim \frac{1}{n}\begin{pmatrix} h^{-1}v_{11} & hv_{12} & v_{13}(a) & h^2v_{14}(a)\\ hv_{12} & hv_{22} & h^2v_{23}(a) & h^4v_{24}(a)\\ v_{13}(a) & h^2v_{23}(a) & h^{-1}v_{33} & hv_{34}\\ h^2v_{14}(a) & h^4v_{24}(a) & hv_{34} & hv_{44}\end{pmatrix}
$$

and then obtain the asymptotic variance of $\widehat{\boldsymbol{\delta}}_{\mathbf{x}}(a)$ by multiplying out $\boldsymbol{\Delta}^{-1}(a)\mathrm{Var}(\boldsymbol{\theta}_n(a)|\mathbf{x})\,\boldsymbol{\Delta}^{-T}(a)$.
$\square$

Case B

In Case B, the terms are all of the same order as in Case A but the constants are different. This means that the results are of the same form as in Case A but with different constants. We need only therefore find the new constants.

For the diagonal block matrices of $\mathbf{\Delta}_n(a)$, for $t > 0$,

$$
\begin{aligned}
n^{-1}\sum_{i=1}^{n}\alpha(\mathbf{x}_i)\,K_{h_j}(x_i^{(j)}-x)^t &\sim h_j^{1-t}\,g_j(x)E\{\alpha(\mathbf{x})|x^{(j)}=x\}\int K_j(z)^t\,dz,\\
n^{-1}\sum_{i=1}^{n}\alpha(\mathbf{x}_i)\,(x_i^{(j)}-x)\,K_{h_j}(x_i^{(j)}-x)^t &\sim h_j^{3-t}\int \partial_j\{\alpha(\mathbf{x})g(\mathbf{x})\}dx_{-j}\int z^2K_j(z)^t\,dz,\\
n^{-1}\sum_{i=1}^{n}\alpha(\mathbf{x}_i)\,(x_i^{(j)}-x)^2\,K_{h_j}(x_i^{(j)}-x)^t &\sim h_j^{3-t}\,g_j(x)E\{\alpha(\mathbf{x})|x^{(j)}=x\}\int z^2K_j(z)^t\,dz,
\end{aligned}
$$

where ∂_j denotes the derivative with respect to $x^{(j)}$ and x_{-j} denotes the component of $\mathbf{x}$ which is not $x^{(j)}$. For the off-diagonal block matrices of $\mathbf{\Delta}_n(a)$,

$$
n^{-1}\sum_{i=1}^{n}\beta(\mathbf{x}_i)\,K_{h_1}(x_i^{(1)}-x)^a\,K_{h_2}(x_i^{(2)}-x)^{1-a} \sim
$$

$$
\begin{cases}
h_1^{1-a}\,h_2^{a}\,\beta(\mathbf{x})g(\mathbf{x})\,\int K_1(z)^a\,dz\,\int K_2(z)^{1-a}\,dz, & 0<a<1,\\
g_1(x)E\{\beta(\mathbf{x})|x^{(1)}=x\}, & a=1,\\
g_2(x)E\{\beta(\mathbf{x})|x^{(2)}=x\}, & a=0;
\end{cases}
$$

$$
n^{-1}\sum_{i=1}^{n}\beta(\mathbf{x}_i)\,K_{h_1}(x_i^{(1)}-x)^a\,K_{h_2}(x_i^{(2)}-x)^{1-a}(x_i^{(1)}-x) \sim
$$

$$
\begin{cases}
h_1^{3-a}\,h_2^{a}\,\partial_1\{\beta(\mathbf{x})g(\mathbf{x})\}\,\int z^2K_1(z)^a\,dz\,\int K_2(z)^{1-a}\,dz, & 0<a<1,\\
h_1^2\int \partial_1\{\beta(x,s)g(x,s)\}ds\,\mu_2^{(1)}, & a=1,\\
g_2(x)E\{\beta(x^{(1)},x)(x^{(1)}-x)|x^{(2)}=x\}, & a=0;
\end{cases}
$$

$$
n^{-1}\sum_{i=1}^{n}\beta(\mathbf{x}_i)\,K_{h_1}(x_i^{(1)}-x)^a\,K_{h_2}(x_i^{(2)}-x)^{1-a}(x_i^{(2)}-x) \sim
$$

$$
\begin{cases}
h_1^{1-a}\,h_2^{2+a}\,\partial_2\{\beta(\mathbf{x})g(\mathbf{x})\}\,\int K_1(z)^a\,dz\,\int z^2K_2(z)^{1-a}\,dz, & 0<a<1,\\
g_1(x)E\{\beta(x,x^{(2)})(x^{(2)}-x)|x^{(1)}=x\}, & a=1,\\
h_2^2\int \partial_2\{\beta(s,x)g(s,x)\}ds\,\mu_2^{(2)}, & a=0;
\end{cases}
$$

$$
n^{-1}\sum_{i=1}^{n}\beta(\mathbf{x}_i)\,K_{h_1}(x_i^{(1)}-x)^a\,K_{h_2}(x_i^{(2)}-x)^{1-a}(x_i^{(1)}-x)\,(x_i^{(2)}-x) \sim
$$

$$\begin{cases} h_1^{3-a}\, h_2^{2+a}\, \partial_{12}\{\beta(\mathbf{x})g(\mathbf{x})\} \int z^2 K_1(z)^a\, dz \int z^2 K_2(z)^{1-a}\, dz, & 0 < a < 1, \\ h_1^2 \int \partial_1\{\beta(x,s)g(x,s)\}(s-x)ds\, \mu_2^{(1)}, & a = 1, \\ h_2^2 \int \partial_2\{\beta(s,x)g(s,x)\}(s-x)ds\, \mu_2^{(2)}, & a = 0. \end{cases}$$

Next, for $\mathbf{t}_n(a)$,

$$n^{-1}\sum_{i=1}^n \alpha(\mathbf{x}_i)\, K_{h_1}(x_i^{(1)}-x)\,(x_i^{(1)}-x)^3 \;\sim\; h_1^4 \int \partial_1\{\alpha(x,s)g(x,s)\}ds\, \mu_4^{(1)},$$

$$n^{-1}\sum_{i=1}^n \alpha(\mathbf{x}_i)\, K_{h_1}(x_i^{(1)}-x)\,(x_i^{(1)}-x)^4 \;\sim\; h_1^4\, g_1(x)E\{\alpha(\mathbf{x})|x^{(1)}=x\}\, \mu_4^{(1)},$$

$$n^{-1}\sum_{i=1}^n \beta(\mathbf{x}_i)\, K_{h_1}(x_i^{(1)}-x)^a\, K_{h_2}(x_i^{(2)}-x)^{1-a}\,(x_i^{(2)}-x)^2 \;\sim$$

$$\begin{cases} h_1^{1-a}\, h_2^{2+a}\, \beta(\mathbf{x})g(\mathbf{x}) \int K_1(z)^a\, dz \int z^2\, K_2(z)^{1-a}\, dz, & 0 < a < 1, \\ g_1(x)\, E\{\beta(x,x^{(2)})(x^{(2)}-x)^2|x^{(1)}=x\}, & a = 1, \\ h_2^2\, g_2(x)\, E\{\beta(x^{(1)},x)|x^{(2)}=x\}\, \mu_2^{(2)}, & a = 0; \end{cases}$$

$$n^{-1}\sum_{i=1}^n \beta(\mathbf{x}_i)\, K_{h_1}(x_i^{(1)}-x)^a\, K_{h_2}(x_i^{(2)}-x)^{1-a}\,(x_i^{(2)}-x)^3 \;\sim$$

$$\begin{cases} h_1^{1-a}\, h_2^{4+a}\, \partial_2\{\beta(\mathbf{x})g(\mathbf{x})\} \int K_1(z)^a\, dz \int z^4\, K_2(z)^{1-a}\, dz, & 0 < a < 1, \\ g_1(x)\, E\{\beta(x,x^{(2)})(x^{(2)}-x)^3|x^{(1)}=x\}, & a = 1, \\ h_2^4 \int \partial_2\{\beta(s,x)g(s,x)\}ds\, \mu_4^{(2)}, & a = 0; \end{cases}$$

$$n^{-1}\sum_{i=1}^n \beta(\mathbf{x}_i)\, K_{h_1}(x_i^{(1)}-x)^a\, K_{h_2}(x_i^{(2)}-x)^{1-a}\,(x_i^{(1)}-x)\,(x_i^{(2)}-x)^2 \;\sim$$

$$\begin{cases} h_1^{3-a}\, h_2^{2+a}\, \partial_1\{\beta(\mathbf{x})g(\mathbf{x})\} \int z^2\, K_1(z)^a\, dz \int z^2\, K_2(z)^{1-a}\, dz, & 0 < a < 1, \\ h_1^2 \int \partial_1\{\beta(x,s)g(x,s)\}(s-x)^2 ds\, \mu_2^{(1)} & a = 1, \\ h_2^2\, g_2(x)\, E\{\beta(x^{(1)},x)(x^{(1)}-x)|x^{(2)}=x\}\, \mu_2^{(2)}, & a = 0; \end{cases}$$

and

$$n^{-1}\sum_{i=1}^n \beta(\mathbf{x}_i)\, K_{h_1}(x_i^{(1)}-x)^a\, K_{h_2}(x_i^{(2)}-x)^{1-a}\,(x_i^{(1)}-x)\,(x_i^{(2)}-x)^3 \;\sim$$

$$\begin{cases} h_1^{3-a}\, h_2^{4+a}\, \partial_{12}\{\beta(\mathbf{x})g(\mathbf{x})\} \int z^2\, K_1(z)^a\, dz \int z^4\, K_2(z)^{1-a}\, dz, & 0 < a < 1, \\ h_1^2 \int \partial_1\{\beta(x,s)g(x,s)\}(s-x)^3 ds\, \mu_2^{(1)}, & a = 1, \\ h_2^4 \int \partial_2\{\beta(s,x)g(s,x)\}(s-x)ds\mu_4^{(2)}, & a = 0. \end{cases}$$

Now consider $\mathbf{\Theta}_n(a)$. For $0 < a < 1$,

$$
\begin{aligned}
(\mathbf{\Theta}_{n11}(a))_{11} &\sim \frac{1}{nh_1} E\{\alpha(\mathbf{x})^2 | x^{(1)} = x\} \sigma_1(x)^2 g_1(x)\, \nu_0^{(1)}, \\
(\mathbf{\Theta}_{n11}(a))_{12} &\sim \frac{h_1}{n} \int \partial_1 \{\alpha(x,s)^2 g(x,s)\} ds\, \sigma_1(x)^2\, \nu_2^{(1)}, \\
(\mathbf{\Theta}_{n11}(a))_{22} &\sim \frac{h_1}{n} E\{\alpha(\mathbf{x})^2 | x^{(1)} = x\} \sigma_1(x)^2 g_1(x)\, \nu_2^{(1)},
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{\Theta}_{n12}(a))_{11} &\sim \frac{1}{n}(h_2/h_1)^{1-a}\, \alpha(\mathbf{x})\beta(\mathbf{x})\sigma_1(x)^2\, g(\mathbf{x}) \int K_1(z)^{2-a}\, dz \int K_2(z)^a\, dz \\
&+ \frac{1}{n}(\alpha(\mathbf{x})\gamma(\mathbf{x}) + \beta(\mathbf{x})^2)\sigma_1(x)\sigma_2(x)\rho(\mathbf{x})\, g(\mathbf{x}) \\
&+ \frac{1}{n}(h_1/h_2)^{1-a}\, \gamma(\mathbf{x})\beta(\mathbf{x})\sigma_2(x)^2\, g(\mathbf{x}) \int K_1(z)^a\, dz \int K_2(z)^{2-a}\, dz, \\
(\mathbf{\Theta}_{n12}(a))_{21} &\sim \frac{h_1 h_2}{n}(h_1/h_2)^a\, \partial_1\{\alpha(\mathbf{x})\beta(\mathbf{x})\sigma_1(x)^2 g(\mathbf{x})\} \int z^2 K_1(z)^{2-a}\, dz \int K_2(z)^a\, dz \\
&+ \frac{h_1^2}{n}\, \partial_1\{(\alpha(\mathbf{x})\gamma(\mathbf{x}) + \beta(\mathbf{x})^2)\sigma_1(x)\rho(\mathbf{x})g(\mathbf{x})\}\, \sigma_2(x)\mu_2^{(1)} \\
&+ \frac{h_1^2}{n}(h_1/h_2)^{1-a}\, \partial_1\{\gamma(\mathbf{x})\beta(\mathbf{x})g(\mathbf{x})\}\, \sigma_2(x)^2 \int z^2 K_1(z)^a\, dz \int K_2(z)^{2-a}\, dz, \\
(\mathbf{\Theta}_{n12}(a))_{12} &\sim \frac{h_2^2}{n}(h_2/h_1)^{1-a}\, \partial_2\{\alpha(\mathbf{x})\beta(\mathbf{x})g(\mathbf{x})\}\, \sigma_1(x)^2 \int K_1(z)^{2-a}\, dz \int z^2 K_2(z)^a\, dz \\
&+ \frac{h_2^2}{n} \partial_2\{(\alpha(\mathbf{x})\gamma(\mathbf{x}) + \beta(\mathbf{x})^2)\sigma_2(x)\rho(\mathbf{x})g(\mathbf{x})\}\, \sigma_1(x)\, \mu_2^{(2)} \\
&+ \frac{h_1 h_2}{n}(h_2/h_1)^a\, \partial_2\{\gamma(\mathbf{x})\beta(\mathbf{x})\sigma_2(x)^2 g(\mathbf{x})\} \int K_1(z)^a\, dz \int z^2 K_2(z)^{2-a}\, dz, \\
(\mathbf{\Theta}_{n12}(a))_{22} &\sim \frac{h_1 h_2^3}{n}(h_1/h_2)^a\, \partial_{12}\{\alpha(\mathbf{x})\beta(\mathbf{x})\sigma_1(x)^2 g(\mathbf{x})\} \int z^2 K_1(z)^{2-a}\, dz \int z^2 K_2(z)^a\, dz \\
&+ \frac{h_1^2 h_2^2}{n} \partial_{12}\{(\alpha(\mathbf{x})\gamma(\mathbf{x}) + \beta(\mathbf{x})^2)\sigma_1(x)\sigma_2(x)\rho(\mathbf{x})g(\mathbf{x})\}\, \mu_2^{(1)}\mu_2^{(2)} \\
&+ \frac{h_1^3 h_2}{n}(h_2/h_1)^a\, \partial_{12}\{\gamma(\mathbf{x})\beta(\mathbf{x})\sigma_2(x)^2 g(\mathbf{x})\} \int z^2 K_1(z)^a\, dz \int z^2 K_2(z)^{2-a}\, dz,
\end{aligned}
$$

where σ_1 is a function of the first component of $\mathbf{x}$ and σ_2 is a function of the second component of $\mathbf{x}$, and

$$
\begin{aligned}
(\mathbf{\Theta}_{n22}(a))_{11} &\sim \frac{1}{n\, h_2} E\{\gamma(\mathbf{x})^2 | x^{(2)} = x\} \sigma_2(x)^2 g_2(x)\, \nu_0^{(2)}, \\
(\mathbf{\Theta}_{n22}(a))_{12} &\sim \frac{h_2}{n} \int \partial_2\{\gamma(s,x)^2 g(s,x)\} ds\, \sigma_2(x)^2\, \nu_2^{(2)}, \\
(\mathbf{\Theta}_{n22}(a))_{22} &\sim \frac{h_2}{n} E\{\gamma(\mathbf{x})^2 | x^{(2)} = x\} \sigma_2(x)^2 g_2(x)\, \nu_2^{(2)}.
\end{aligned}
$$

□

Appendix B. Proof of Theorem 4

Case A

We can write $\boldsymbol{\Delta} = \boldsymbol{\Delta}_{11} \otimes \overline{\mathbf{W}}$ so that $\boldsymbol{\Delta}^{-1} = \boldsymbol{\Delta}_{11}^{-1} \otimes \overline{\mathbf{W}}^{-1}$. We have

$$\begin{aligned}
\mathrm{Var}(\boldsymbol{\theta}_n(a)|x) \quad &\sim \quad \frac{g(x)}{nh} \begin{pmatrix} \nu_0 & h^2\,\nu_2\,g'(x)/g(x) \\ h^2\,\nu_2\,g'(x)/g(x) & h^2\,\nu_2 \end{pmatrix} \otimes \frac{1}{n}\sum_{i=1}^{n} \mathbf{W}_i \boldsymbol{\Sigma}_i \mathbf{W}_i \\
&= \quad \frac{g(x)}{nh} \begin{pmatrix} \nu_0 & h^2\,\nu_2\,g'(x)/g(x) \\ h^2\,\nu_2\,g'(x)/g(x) & h^2 \nu_2 \end{pmatrix} \otimes \overline{\mathbf{W}}
\end{aligned}$$

so the result obtains.

Case B

For the diagonal block matrices of $\boldsymbol{\Delta}_n$, we can apply standard arguments to show that

$$\begin{aligned}
n^{-1}\sum_{i=1}^{n} \alpha(x_i)\,K_h(x_i - x) \quad &\sim \quad \alpha(x)\,g(x), \\
n^{-1}\sum_{i=1}^{n} \alpha(x_i)\,(x_i - x)\,K_h(x_i - x) \quad &\sim \quad h^2\,\{\alpha(x)g'(x) + \alpha'(x)g(x)\}\,\mu_2 \\
n^{-1}\sum_{i=1}^{n} \alpha(x_i)\,(x_i - x)^2\,K_h(x_i - x) \quad &\sim \quad h^2\,\alpha(x)\,g(x)\,\mu_2.
\end{aligned}$$

The limit of $\boldsymbol{\Delta}_n$ is therefore

$$\boldsymbol{\Delta} \quad = \quad \begin{pmatrix} \boldsymbol{\Delta}_\alpha & \boldsymbol{\Delta}_\beta \\ \boldsymbol{\Delta}_\beta & \boldsymbol{\Delta}_\gamma \end{pmatrix},$$

where

$$\boldsymbol{\Delta}_\eta \quad = \quad \begin{pmatrix} \eta(x)g(x) & h^2\{\eta(x)g'(x) + \eta'(x)g(x)\}\mu_2 \\ h^2\{\eta(x)g'(x) + \eta'(x)g(x)\}\mu_2 & h^2\eta(x)g(x)\mu_2 \end{pmatrix}.$$

In general, if we write

$$\boldsymbol{\Delta} \quad = \quad \begin{pmatrix} d_{11} & h^2\,d_{12} & d_{13} & h^2\,d_{14} \\ h^2\,d_{12} & h^2\,d_{22} & h^2\,d_{14} & h^2\,d_{24} \\ d_{13} & h^2\,d_{14} & d_{33} & h^2\,d_{34} \\ h^2\,d_{14} & h^2\,d_{24} & h^2\,d_{34} & h^2\,d_{44} \end{pmatrix},$$

then we can show that

$$\boldsymbol{\Delta}^{-1} \sim \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} & \delta_{14} \\ \delta_{12} & h^{-2}\,\delta_{22} & \delta_{23} & h^{-2}\,\delta_{24} \\ \delta_{13} & \delta_{23} & \delta_{33} & \delta_{34} \\ \delta_{14} & h^{-2}\,\delta_{24} & \delta_{34} & h^{-2}\,\delta_{44} \end{pmatrix}.$$

Using the fact that $\mathbf{W}(x)^{-1} = \boldsymbol{\Sigma}(x)$ and $|\mathbf{W}(x)| = \alpha(x)\gamma(x) - \beta(x)^2$, we have

$$\frac{1}{|\mathbf{W}(x)|}\begin{pmatrix} \gamma(x) & -\beta(x) \\ -\beta(x) & \alpha(x) \end{pmatrix} = \begin{pmatrix} \sigma_1(x)^2 & \sigma_1(x)\sigma_2(x)\rho(x) \\ \sigma_1(x)\sigma_2(x)\rho(x) & \sigma_2(x)^2 \end{pmatrix}.$$

Moreover, writing $\partial f(x) = f'(x)$, we have

$$\partial\sigma_1(x)^2 = -\{\gamma'(x)\beta(x)^2 + \alpha'(x)\gamma(x)^2 - 2\gamma(x)\beta'(x)\beta(x)\}/|\mathbf{W}(x)|^2,$$

$$\partial\sigma_2(x)^2 = -\{\alpha'(x)\beta(x)^2 + \gamma'(x)\alpha(x)^2 - 2\alpha(x)\beta'(x)\beta(x)\}/|\mathbf{W}(x)|^2$$

and

$$\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\} = -\{\beta'(x)\alpha(x)\gamma(x) - \alpha'(x)\gamma(x)\beta(x) - \beta(x)\gamma'(x)\alpha(x) + \beta'(x)\beta(x)^2\}/|\mathbf{W}(x)|^2.$$

We can then write

$$\begin{aligned} |\boldsymbol{\Delta}|/h^4 &= d_{24}^2 d_{13}^2 - d_{24}^2 d_{11} d_{33} - d_{13}^2 d_{22} d_{44} + d_{11} d_{22} d_{33} d_{44} \\ &= \{\beta(x)g(x)\mu_2\}^2\{\beta(x)g(x)\}^2 - \{\beta(x)g(x)\mu_2\}^2\alpha(x)g(x)\gamma(x)g(x) \\ &\quad -\{\beta(x)g(x)\}^2\alpha(x)g(x)\mu_2\gamma(x)g(x)\mu_2 + \alpha(x)g(x)\alpha(x)g(x)\mu_2\gamma(x)g(x)\gamma(x)g(x)\mu_2 \\ &= \beta(x)^4 g(x)^4\mu_2^2 - 2\beta(x)^2\alpha(x)\gamma(x)g(x)^4\mu_2^2 + \alpha(x)^2\gamma(x)^2 g(x)^4\mu_2^2 \\ &= g(x)^4\mu_2^2\{\beta(x)^4 - 2\beta(x)^2\alpha(x)\gamma(x) + \alpha(x)^2\gamma(x)^2\} \\ &= g(x)^4\mu_2^2\{\alpha(x)\gamma(x) - \beta(x)^2\}^2 \\ &= g(x)^4\mu_2^2|\mathbf{W}(x)|^2 \end{aligned}$$

so

$$\begin{aligned} |\boldsymbol{\Delta}|\delta_{11}/h^4 &= d_{22}d_{33}d_{44} - d_{24}^2 d_{33} \\ &= \alpha(x)g(x)\mu_2\gamma(x)g(x)\gamma(x)g(x)\mu_2 - \{\beta(x)g(x)\mu_2\}^2\gamma(x)g(x) \\ &= \mu_2^2 g(x)^3\gamma(x)\{\alpha(x)\gamma(x) - \beta(x)^2\} \\ &= \mu_2^2 g(x)^3\sigma_1(x)^2|\mathbf{W}(x)|^2, \end{aligned}$$

$$
\begin{aligned}
|\boldsymbol{\Delta}|\delta_{12}/h^4 &= -d_{12}d_{33}d_{44} + d_{24}d_{14}d_{33} + d_{14}d_{13}d_{44} - d_{24}d_{13}d_{34} \\
&= -\mu_2\{\alpha(x)g'(x) + \alpha'(x)g(x)\}\gamma(x)g(x)\gamma(x)g(x)\mu_2 \\
&\quad +\beta(x)g(x)\mu_2\mu_2\{\beta(x)g'(x) + \beta'(x)g(x)\}\gamma(x)g(x) \\
&\quad +\mu_2\{\beta(x)g'(x) + \beta'(x)g(x)\}\beta(x)g(x)\gamma(x)g(x)\mu_2 \\
&\quad -\beta(x)g(x)\mu_2\beta(x)g(x)\mu_2\{\gamma(x)g'(x) + \gamma'(x)g(x)\} \\
&= -\mu_2^2 g(x)^2[\gamma(x)^2\{\alpha(x)g'(x) + \alpha'(x)g(x)\} - \beta(x)\gamma(x)\{\beta(x)g'(x) + \beta'(x)g(x)\} \\
&\quad -\beta(x)\gamma(x)\{\beta(x)g'(x) + \beta'(x)g(x)\} + \beta(x)^2\{\gamma(x)g'(x) + \gamma'(x)g(x)\}] \\
&= -\mu_2^2 g(x)^2[g'(x)\gamma(x)\{\alpha(x)\gamma(x) - \beta(x)^2\} \\
&\quad +g(x)\{\gamma(x)^2\alpha'(x) - 2\beta(x)\gamma(x)\beta'(x) + \beta(x)^2\gamma'(x)\}] \\
&= -\mu_2^2 g(x)^2|\mathbf{W}(x)|^2[g'(x)\sigma_1(x)^2 - g(x)\partial\sigma_1(x)^2],
\end{aligned}
$$

$$
\begin{aligned}
|\boldsymbol{\Delta}|\delta_{13}/h^4 &= d_{24}^2 d_{13} - d_{22}d_{13}d_{44} \\
&= \{\beta(x)g(x)\mu_2\}^2\beta(x)g(x) - \alpha(x)g(x)\mu_2\beta(x)g(x)\gamma(x)g(x)\mu_2 \\
&= -\mu_2^2 g(x)^3\beta(x)\{\alpha(x)\gamma(x) - \beta(x)^2\} \\
&= \mu_2^2 g(x)^3|\mathbf{W}(x)|^2\sigma_1(x)\sigma_2(x)\rho(x)
\end{aligned}
$$

and

$$
\begin{aligned}
|\boldsymbol{\Delta}|\delta_{14}/h^4 &= -d_{14}d_{13}d_{24} + d_{12}d_{24}d_{33} + d_{22}d_{13}d_{34} - d_{22}d_{14}d_{33} \\
&= -\mu_2\{\beta(x)g'(x) + \beta'(x)g(x)\}\beta(x)g(x)\beta(x)g(x)\mu_2 \\
&\quad +\mu_2\{\alpha(x)g'(x) + \alpha'(x)g(x)\}\beta(x)g(x)\mu_2\gamma(x)g(x) \\
&\quad +\alpha(x)g(x)\mu_2\beta(x)g(x)\mu_2\{\gamma(x)g'(x) + \gamma'(x)g(x)\} \\
&\quad -\alpha(x)g(x)\mu_2\mu_2\{\beta(x)g'(x) + \beta'(x)g(x)\}\gamma(x)g(x) \\
&= -\mu_2^2 g(x)^2[\beta(x)^2\{\beta(x)g'(x) + \beta'(x)g(x)\} - \beta(x)\gamma(x)\{\alpha(x)g'(x) + \alpha'(x)g(x)\} \\
&\quad -\alpha(x)\beta(x)\{\gamma(x)g'(x) + \gamma'(x)g(x)\} + \alpha(x)\gamma(x)\{\beta(x)g'(x) + \beta'(x)g(x)\}] \\
&= \mu_2^2 g(x)^2[g'(x)\beta(x)\{\alpha(x)\gamma(x) - \beta(x)^2\} \\
&\quad -g(x)\{\beta(x)^2\beta'(x) - \beta(x)\gamma(x)\alpha'(x) - \alpha(x)\beta(x)\gamma'(x) + \alpha(x)\gamma(x)\beta'(x)\}] \\
&= -\mu_2^2 g(x)^2|\mathbf{W}(x)|^2[g'(x)\sigma_1(x)\sigma_2(x)\rho(x) + g(x)\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}].
\end{aligned}
$$

From the second row,

$$
\begin{aligned}
|\mathbf{\Delta}|\delta_{22}/h^4 &= -d_{13}^2 d_{44} + d_{11}d_{33}d_{44} \\
&= -\{\beta(x)g(x)\}^2\gamma(x)g(x)\mu_2 + \alpha(x)g(x)\gamma(x)g(x)\gamma(x)g(x)\mu_2 \\
&= \mu_2 g(x)^3\gamma(x)\{\alpha(x)\gamma(x) - \beta(x)^2\} \\
&= |\mathbf{\Delta}|\delta_{11}/(h^4\mu_2),
\end{aligned}
$$

$$
\begin{aligned}
|\mathbf{\Delta}|\delta_{23}/h^4 &= -d_{11}d_{14}d_{44} + d_{11}d_{24}d_{34} + d_{12}d_{13}d_{44} - d_{14}d_{13}d_{24} \\
&= -\alpha(x)g(x)\mu_2\{\beta(x)g'(x) + \beta'(x)g(x)\}\gamma(x)g(x)\mu_2 \\
&\quad +\alpha(x)g(x)\beta(x)g(x)\mu_2\mu_2\{\gamma(x)g'(x) + \gamma'(x)g(x)\} \\
&\quad +\mu_2\{\alpha(x)g'(x) + \alpha'(x)g(x)\}\beta(x)g(x)\gamma(x)g(x)\mu_2 \\
&\quad -\mu_2\{\beta(x)g'(x) + \beta'(x)g(x)\}\beta(x)g(x)\beta(x)g(x)\mu_2 \\
&= -\mu_2^2 g(x)^2[\alpha(x)\gamma(x)\{\beta(x)g'(x) + \beta'(x)g(x)\} - \alpha(x)\beta(x)\{\gamma(x)g'(x) + \gamma'(x)g(x)\} \\
&\quad -\beta(x)\gamma(x)\{\alpha(x)g'(x) + \alpha'(x)g(x)\} + \beta(x)^2\{\beta(x)g'(x) + \beta'(x)g(x)\}] \\
&= -\mu_2^2 g(x)^2[g'(x)\beta(x)\{\alpha(x)\gamma(x) - \beta(x)^2\} \\
&\quad -g(x)\{\alpha(x)\gamma(x)\beta'(x) - \alpha(x)\beta(x)\gamma'(x) - \beta(x)\gamma(x)\alpha'(x) + \beta(x)^2\beta'(x)\}] \\
&= -|\mathbf{\Delta}|\delta_{14}/h^4
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathbf{\Delta}|\delta_{24}/h^4 &= -d_{11}d_{24}d_{33} + d_{13}^2 d_{24} \\
&= -\alpha(x)g(x)\beta(x)g(x)\mu_2\gamma(x)g(x) + \{\beta(x)g(x)\}^2\beta(x)g(x)\mu_2 \\
&= -\mu_2 g(x)^3\beta(x)\{\alpha(x)\gamma(x) - \beta(x)^2\} \\
&= |\mathbf{\Delta}|\delta_{13}/(h^4\mu_2).
\end{aligned}
$$

Finally, from the third and fourth rows,

$$
\begin{aligned}
|\mathbf{\Delta}|\delta_{33}/h^4 &= d_{11}d_{22}d_{44} - d_{11}d_{24}^2 \\
&= \alpha(x)g(x)\alpha(x)g(x)\mu_2\gamma(x)g(x)\mu_2 - \alpha(x)g(x)\{\beta(x)g(x)\mu_2\}^2 \\
&= \mu_2^2 g(x)^3\alpha(x)\{\alpha(x)\gamma(x) - \beta(x)^2\} \\
&= \mu_2^2 g(x)^3\sigma_2(x)^2|\mathbf{W}(x)|^2,
\end{aligned}
$$

$$
\begin{aligned}
|\mathbf{\Delta}|\delta_{34}/h^4 &= -d_{11}d_{22}d_{34} + d_{11}d_{24}d_{14} - d_{12}d_{13}d_{24} + d_{14}d_{13}d_{22} \\
&= -\alpha(x)g(x)\alpha(x)g(x)\mu_2\mu_2\{\gamma(x)g'(x) + \gamma'(x)g(x)\} \\
&\quad +\alpha(x)g(x)\beta(x)g(x)\mu_2\mu_2\{\beta(x)g'(x) + \beta'(x)g(x)\} \\
&\quad -\mu_2\{\alpha(x)g'(x) + \alpha'(x)g(x)\}\beta(x)g(x)\beta(x)g(x)\mu_2 \\
&\quad +\mu_2\{\beta(x)g'(x) + \beta'(x)g(x)\}\beta(x)g(x)\alpha(x)g(x)\mu_2 \\
&= -\mu_2^2 g(x)^2[\alpha(x)^2\{\gamma(x)g'(x) + \gamma'(x)g(x)\} \\
&\quad -2\alpha(x)\beta(x)\{\beta(x)g'(x) + \beta'(x)g(x)\} \\
&\quad +\beta(x)^2\{\alpha(x)g'(x) + \alpha'(x)g(x)\} \\
&= -\mu_2^2 g(x)^2[g'(x)\alpha(x)\{\alpha(x)\gamma(x) - \beta(x)^2\} \\
&\quad +g(x)\{\alpha(x)^2\gamma'(x) - 2\alpha(x)\beta(x)\beta'(x) + \beta(x)^2\alpha'(x)\}] \\
&= -\mu_2^2 g(x)^2|\mathbf{W}(x)|^2\{g'(x)\sigma_2(x)^2 - g(x)\partial\sigma_2(x)^2\}
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathbf{\Delta}|\delta_{44}/h^4 &= d_{11}d_{22}d_{33} - d_{22}d_{13}^2 \\
&= \alpha(x)g(x)\alpha(x)g(x)\mu_2\gamma(x)g(x) - \alpha(x)g(x)\mu_2(\beta(x)g(x))^2 \\
&= \mu_2 g(x)^3\alpha(x)\{\alpha(x)\gamma(x) - \beta(x)^2\} \\
&= |\mathbf{\Delta}|\delta_{33}/(h^4\mu_2).
\end{aligned}
$$

Now

$$
n^{-1}\sum_{i=1}^{n}\alpha(x_i)\,K_h(x_i - x)\,(x_i - x)^3 \;\sim\; h^4\,\{\alpha(x)g'(x) + \alpha'(x)g(x)\}\,\mu_4,
$$

$$
n^{-1}\sum_{i=1}^{n}\alpha(x_i)\,K_h(x_i - x)\,(x_i - x)^4 \;\sim\; h^4\,\alpha(x)g(x)\,\mu_4
$$

and similarly for the other terms, so that

$$\mathbf{t} \sim \begin{pmatrix} h^2\,\mu_2\,g(x)\,\{f_1''(x)\alpha(x) + f_2''(x)\beta(x)\}/2 \\ \\ h^4\,\mu_4\,[\{\alpha(x)g'(x) + \alpha'(x)g(x)\}f_1''(x)/2 + \alpha(x)g(x)f_1'''(x)/3! \\ +\{\beta(x)g'(x) + \beta'(x)g(x)\}f_2''(x)/2 + \beta(x)g(x)f_2'''(x)/3!] \\ \\ h^2\,\mu_2\,g(x)\,\{f_1''(x)\beta(x) + f_2''(x)\gamma(x)\}/2 \\ \\ h^4\,\mu_4\,[\{\beta(x)g'(x) + \beta'(x)g(x)\}f_1''(x)/2 + \beta(x)g(x)f_1'''(x)/3! \\ +\{\gamma(x)g'(x) + \gamma'(x)g(x)\}f_2''(x)/2 + \gamma(x)g(x)f_2'''(x)/3!] \end{pmatrix}.$$

Writing $\mathbf{t} = (h^2t_1, h^4t_2, h^2t_3, h^4t_4)^T$, the asymptotic bias is

$$\mathbf{\Delta}^{-1}\,\mathbf{t} \sim h^2 \begin{pmatrix} \delta_{11}t_1 + \delta_{13}t_3 \\ \delta_{12}t_1 + \delta_{22}t_2 + \delta_{23}t_3 + \delta_{24}t_4 \\ \delta_{13}t_1 + \delta_{33}t_3 \\ \delta_{14}t_1 + \delta_{24}t_2 + \delta_{34}t_3 + \delta_{44}t_4 \end{pmatrix}.$$

The bias calculation for estimating f_1 is straightforward. We obtain

$$\begin{aligned} |\mathbf{\Delta}|(\delta_{11}t_1 + \delta_{13}t_3)/h^4 &= \mu_2^2 g(x)^3\gamma(x)|\mathbf{W}(x)|\mu_2\,g(x)\,\{f_1''(x)\alpha(x) + f_2''(x)\beta(x)\}/2 \\ &\quad -\mu_2^2 g(x)^3\beta(x)|\mathbf{W}(x)|\mu_2 g(x)\{f_2''(x)\gamma(x) + f_1''(x)\beta(x)\}/2 \\ &= \mu_2^3 g(x)^4|\mathbf{W}(x)|[\gamma(x)\{f_1''(x)\alpha(x) + f_2''(x)\beta(x)\} \\ &\quad -\beta(x)\{f_2''(x)\gamma(x) + f_1''(x)\beta(x)\}]/2 \\ &= \mu_2^3 g(x)^4|\mathbf{W}(x)|^2 f_1''(x)/2 \end{aligned}$$

so that

$$\begin{aligned} \delta_{11}t_1 + \delta_{13}t_3 &= h^4\mu_2^3 g(x)^4|\mathbf{W}(x)|^2 f_1''(x)/2|\mathbf{\Delta}| \\ &= \mu_2^3 g(x)^4|\mathbf{W}(x)|^2 f_1''(x)/2g(x)^4\mu_2^2|\mathbf{W}(x)|^2 \\ &= \mu_2 f_1''(x)/2. \end{aligned}$$

The bias calculation for estimating f_1' is more complicated. We obtain

$|\mathbf{\Delta}|(\delta_{21}t_1 + \delta_{22}t_2 + \delta_{23}t_3 + \delta_{24}t_4)/h^4 =$

$$
\begin{aligned}
&-\mu_2^3 g(x)^3|\mathbf{W}(x)|[g'(x)\gamma(x)-|\mathbf{W}(x)|g(x)\partial\sigma_1(x)^2]\{f_1''(x)\alpha(x)+f_2''(x)\beta(x)\}/2\\
&+\mu_2\mu_4 g(x)^3\gamma(x)|\mathbf{W}(x)|\,[\{\alpha(x)g'(x)+\alpha'(x)g(x)\}f_1''(x)/2+\alpha(x)g(x)f_1'''(x)/3!\\
&+\{\beta(x)g'(x)+\beta'(x)g(x)\}f_2''(x)/2+\beta(x)g(x)f_2'''(x)/3!]\\
&+\mu_2^3 g(x)^3|\mathbf{W}(x)|[g'(x)\beta(x)+|\mathbf{W}(x)|g(x)\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}]\{f_1''(x)\beta(x)+f_2''(x)\gamma(x)\}/2\\
&-\mu_2\mu_4 g(x)^3\beta(x)|\mathbf{W}(x)|\,[\{\beta(x)g'(x)+\beta'(x)g(x)\}f_1''(x)/2+\beta(x)g(x)f_1'''(x)/3!\\
&+\{\gamma(x)g'(x)+\gamma'(x)g(x)\}f_2''(x)/2+\gamma(x)g(x)f_2'''(x)/3!]\\
=\;&-\mu_2^3 g(x)^3 g'(x)|\mathbf{W}(x)|^2 f_1''(x)/2\\
&+\mu_2^3 g(x)^4|\mathbf{W}(x)|^2[\alpha(x)\partial\sigma_1(x)^2+\beta(x)\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}]f_1''(x)/2\\
&+\mu_2\mu_4 g(x)^3 g'(x)|\mathbf{W}(x)|^2 f_1''(x)/2\\
&+\mu_2\mu_4 g(x)^4|\mathbf{W}(x)|\,\{\alpha'(x)\gamma(x)-\beta'(x)\beta(x)\}f_1''(x)/2\\
&+\mu_2^3 g(x)^4|\mathbf{W}(x)|^2[\beta(x)\partial\sigma_1(x)^2+\gamma(x)\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}]f_2''(x)/2\\
&+\mu_2\mu_4 g(x)^4|\mathbf{W}(x)|\,\{\gamma(x)\beta'(x)+\beta(x)\gamma'(x)\}f_2''(x)/2\\
&+\mu_2\mu_4 g(x)^4|\mathbf{W}(x)|^2 f_1'''(x)/3!\\
=\;&(\mu_4-\mu_2^2)\mu_2 g(x)^3 g'(x)|\mathbf{W}(x)|^2 f_1''(x)/2+\mu_2\mu_4 g(x)^4|\mathbf{W}(x)|^2 f_1'''(x)/3!\\
&+\mu_2^3 g(x)^4|\mathbf{W}(x)|^2\Big\{[\alpha(x)\partial\sigma_1(x)^2+\beta(x)\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}]f_1''(x)/2\\
&+[\beta(x)\partial\sigma_1(x)^2+\gamma(x)\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}]f_2''(x)/2\Big\}\\
&+\mu_2\mu_4 g(x)^4|\mathbf{W}(x)|\,[\{\alpha'(x)\gamma(x)-\beta'(x)\beta(x)\}f_1''(x)/2+\{\gamma(x)\beta'(x)+\beta(x)\gamma'(x)\}f_2''(x)/2]
\end{aligned}
$$

so that

$$
\begin{aligned}
&\delta_{21}t_1+\delta_{22}t_2+\delta_{23}t_3+\delta_{24}t_4=(\mu_4-\mu_2^2)g'(x)f_1''(x)/(2g(x)\mu_2)+\mu_4 f_1'''(x)/(\mu_2 3!)\\
&\quad+\mu_2\Big\{[\alpha(x)\partial\sigma_1(x)^2+\beta(x)\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}]f_1''(x)/2\\
&\quad+[\beta(x)\partial\sigma_1(x)^2+\gamma(x)\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}]f_2''(x)/2\Big\}\\
&\quad+\mu_4/(\mu_2|\mathbf{W}(x)|)\,[\{\alpha'(x)\gamma(x)-\beta'(x)\beta(x)\}f_1''(x)/2+\{\gamma(x)\beta'(x)+\beta(x)\gamma'(x)\}f_2''(x)/2].
\end{aligned}
$$

Similar arguments show that that

$$
\begin{aligned}
&\delta_{14}t_1+\delta_{24}t_2+\delta_{34}t_3+\delta_{44}t_4=(\mu_4-\mu_2^2)g'(x)f_2''(x)/(2g(x)\mu_2)+\mu_4 f_2'''(x)/(\mu_2 3!)\\
&\quad+\mu_2\Big\{[\alpha(x)\partial\sigma_1(x)^2-\beta(x)\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}]f_2''(x)/2
\end{aligned}
$$

$$- \, [\beta(x)\partial\sigma_2(x)^2 + \gamma(x)\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}]f_2''(1)/2\Big\}$$
$$+\mu_4/(\mu_2|\mathbf{W}(x)|)\,[\{\alpha(x)\gamma'(x) - \beta'(x)\beta(x)\}f_2''(x)/2 + \,\{\alpha(x)\beta'(x) + \beta(x)\alpha'(x)\}f_1''(x)/2].$$

Now consider $\boldsymbol{\Theta}_n$. We have

$$\begin{aligned} n^2\,\boldsymbol{\Theta}_{n11} &= \mathbf{X}_x^T\mathbf{K}^2\,\mathrm{diag}(\alpha(x_i))\,\mathbf{X}_x, \\ n^2\,\boldsymbol{\Theta}_{n12} &= \mathbf{X}_x^T\mathbf{K}^2\,\mathrm{diag}(\beta(x_i))\,\mathbf{X}_x \\ n^2\,\boldsymbol{\Theta}_{n22} &= \mathbf{X}_x^T\mathbf{K}^2\,\mathrm{diag}(\gamma(x_i))\,\mathbf{X}_x. \end{aligned}$$

Now, it follows that

$$\boldsymbol{\Theta}_{11} \sim \frac{1}{n}\begin{pmatrix} \frac{1}{h}\alpha(x)g(x)\nu_0 & h\{\alpha(x)g'(x)+\alpha'(x)g(x)\}\nu_2 \\ h\{\alpha(x)g'(x)+\alpha'(x)g(x)\}\nu_2 & h\alpha(x)g(x)\nu_2 \end{pmatrix}$$

so we can write

$$\mathrm{Var}(\boldsymbol{\theta}_n|x) \;\sim\; \frac{1}{n}\begin{pmatrix} h^{-1}v_{11} & hv_{12} & h^{-1}v_{13} & hv_{14} \\ hv_{12} & hv_{22} & hv_{14} & hv_{24} \\ h^{-1}v_{13} & hv_{14} & h^{-1}v_{33} & hv_{34} \\ hv_{14} & hv_{24} & hv_{34} & hv_{44} \end{pmatrix}.$$

We obtain the asymptotic variance of $\widehat{\boldsymbol{\delta}}_x$ by multiplying out $\boldsymbol{\Delta}^{-1}\mathrm{Var}(\boldsymbol{\theta}_n|x)\,\boldsymbol{\Delta}^{-T}$ to obtain

$$\begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} & \delta_{14} \\ \delta_{12} & \frac{\delta_{11}}{h^2\mu_2} & \delta_{14} & \frac{\delta_{13}}{h^2\mu_2} \\ \delta_{13} & \delta_{14} & \delta_{33} & \delta_{34} \\ \delta_{14} & \frac{\delta_{13}}{h^2\mu_2} & \delta_{34} & \frac{\delta_{33}}{h^2\mu_2} \end{pmatrix}\begin{pmatrix} h^{-1}v_{11} & hv_{12} & h^{-1}v_{13} & hv_{14} \\ hv_{12} & hv_{22} & hv_{14} & hv_{24} \\ h^{-1}v_{13} & hv_{14} & h^{-1}v_{33} & hv_{34} \\ hv_{14} & hv_{24} & hv_{34} & hv_{44} \end{pmatrix}\begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} & \delta_{14} \\ \delta_{12} & \frac{\delta_{11}}{h^2\mu_2} & \delta_{14} & \frac{\delta_{13}}{h^2\mu_2} \\ \delta_{13} & \delta_{14} & \delta_{33} & \delta_{34} \\ \delta_{14} & \frac{\delta_{13}}{h^2\mu_2} & \delta_{34} & \frac{\delta_{33}}{h^2\mu_2} \end{pmatrix}.$$

It is convenient to calculate two 2×2 submatrices first because we can exploit the relationship between them. The first of these is

$$\mathrm{Var}\begin{pmatrix} \widehat{f}_1(x) \\ \widehat{f}_2(x) \end{pmatrix} = \begin{pmatrix} c_{11} & c_{13} \\ c_{13} & c_{33} \end{pmatrix},$$

where

$$\begin{aligned} |\mathbf{W}(x)|^2 nhc_{11} &= |\mathbf{W}(x)|^2(\delta_{11}^2 v_{11} + 2\delta_{11}\delta_{13}v_{13} + \delta_{13}^2 v_{33}) \\ &= (\frac{\gamma(x)}{g(x)})^2\alpha(x)g(x)\nu_0 - 2\frac{\gamma(x)}{g(x)}\frac{\beta(x)}{g(x)}\beta(x)g(x)\nu_0 + (\frac{\beta(x)}{g(x)})^2\gamma(x)g(x)\nu_0 \end{aligned}$$

$$
\begin{aligned}
&= \frac{\nu_0}{g(x)}\{\alpha(x)\gamma(x)^2 - \gamma(x)\beta(x)^2\} \\
&= \frac{\nu_0}{g(x)}\gamma(x)|\mathbf{W}(x)| \\
&= \frac{\nu_0}{g(x)}\sigma_1(x)^2|\mathbf{W}(x)|^2,
\end{aligned}
$$

$$
\begin{aligned}
|\mathbf{W}(x)|^2 nhc_{13} &= |\mathbf{W}(x)|^2(\delta_{13}\delta_{11}v_{11} + \delta_{33}\delta_{13}v_{33} + \delta_{13}^2 v_{13} + \delta_{33}\delta_{11}\nu_{13}) \\
&= -\frac{\beta(x)\gamma(x)}{g(x)^2}\alpha(x)g(x)\nu_0 - \frac{\alpha(x)\beta(x)}{g(x)^2}\gamma(x)g(x)\nu_0 \\
&\quad +(\frac{\beta(x)}{g(x)})^2\beta(x)g(x)\nu_0 + \frac{\alpha(x)\gamma(x)}{g(x)^2}\beta(x)g(x)\nu_0 \\
&= \frac{\nu_0}{g(x)}\{-\alpha(x)\beta(x)\gamma(x) - \beta(x)^3\} \\
&= -\frac{\nu_0}{g(x)}\beta(x)|\mathbf{W}(x)| \\
&= \frac{\nu_0}{g(x)}\sigma_1(x)\sigma_2(x)\rho(x)|\mathbf{W}(x)|^2
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathbf{W}(x)|^2 nhc_{33} &= |\mathbf{W}(x)|^2(\delta_{13}^2 v_{11} + 2\delta_{13}\delta_{33}v_{13} + \delta_{33}^2 v_{33}) \\
&= (\frac{\beta(x)}{g(x)})^2\alpha(x)g(x)\nu_0 - 2\frac{\beta(x)}{g(x)}\frac{\alpha(x)}{g(x)}\beta(x)g(x)\nu_0 + (\frac{\alpha(x)}{g(x)})^2\gamma(x)g(x)\nu_0 \\
&= \frac{\nu_0}{g(x)}\{-\alpha(x)\beta(x)^2 + \gamma(x)\alpha(x)^2\} \\
&= \frac{\nu_0}{g(x)}\alpha(x)|\mathbf{W}(x)| \\
&= \frac{\nu_0}{g(x)}\sigma_2^2(x)|\mathbf{W}(x)|^2.
\end{aligned}
$$

It follows that

$$
\mathrm{Var}\begin{pmatrix} \widehat{f}_1(x) \\ \widehat{f}_2(x) \end{pmatrix} = \frac{\nu_0}{nhg(x)}\mathbf{\Sigma}(x).
$$

The second submatrix is

$$
\mathrm{Var}\begin{pmatrix} \widehat{f}_1'(x) \\ \widehat{f}_2'(x) \end{pmatrix} = \begin{pmatrix} c_{22} & c_{24} \\ c_{24} & c_{44} \end{pmatrix}.
$$

The terms are the same as those above with h replaced by $h^3\mu_2^2$ and ν_0 replaced by ν_2 so

$$
\mathrm{Var}\begin{pmatrix} \widehat{f}_1(x) \\ \widehat{f}_2(x) \end{pmatrix} = \frac{\nu_2}{nh^3 g(x)\mu_2^2}\mathbf{W}(x)^{-1} = \frac{\nu_2}{nh^3 g(x)\mu_2^2}\mathbf{\Sigma}(x).
$$

There are four remaining covariance terms but two of these can be derived from the others so we only require two further terms to be calculated explicitly. First, we have

$$
\begin{aligned}
nh|\mathbf{W}(x)|^2 c_{12} &= nh|\mathbf{W}(x)|^2 \mathrm{Cov}(\widehat{f}_1(x), \widehat{f}_1'(x)) \\
&= |\mathbf{W}(x)|^2[\delta_{11}\delta_{12}(v_{11}+v_{22}/\mu_2)+\delta_{12}\delta_{13}(v_{13}+v_{24}/\mu_2)+\delta_{11}\delta_{14}(v_{13}+v_{24}/\mu_2) \\
&\quad +\delta_{14}\delta_{13}(v_{33}+v_{44}/\mu_2)+\delta_{11}^2 v_{12}/\mu_2+2\delta_{11}\delta_{13}v_{14}/\mu_2+\delta_{13}^2 v_{34}/\mu_2] \\
&= |\mathbf{W}(x)|^2[\delta_{11}\delta_{12}\alpha(x)g(x)(\nu_0+\nu_2/\mu_2)+(\delta_{12}\delta_{13}+\delta_{11}\delta_{14})\beta(x)g(x)(\nu_0+\nu_2/\mu_2) \\
&\quad +\delta_{14}\delta_{13}\gamma(x)g(x)(\nu_0+\nu_2/\mu_2)+\delta_{11}^2\{\alpha(x)g'(x)+\alpha'(x)g(x)\}\nu_2/\mu_2 \\
&\quad +2\delta_{11}\delta_{13}\{\beta(x)g'(x)+\beta'(x)g(x)\}\nu_2/\mu_2+\delta_{13}^2\{\gamma(x)g'(x)+\gamma'(x)g(x)\}\nu_2/\mu_2] \\
&= \gamma(x)|\mathbf{W}(x)|\delta_{12}\alpha(x)(\nu_0+\nu_2/\mu_2) \\
&\quad +|\mathbf{W}(x)|\{-\delta_{12}\beta(x)+\gamma(x)\delta_{14}\}\beta(x)g(x)(\nu_0+\nu_2/\mu_2) \\
&\quad -\delta_{14}\beta(x)|\mathbf{W}(x)|\gamma(x)(\nu_0+\nu_2/\mu_2) \\
&\quad +(\nu_2/\mu_2 g(x)^2)[\gamma(x)^2\{\alpha(x)g'(x)+\alpha'(x)g(x)\} \\
&\quad -2\gamma(x)\beta(x)\{\beta(x)g'(x)+\beta'(x)g(x)\}+\beta(x)^2\{\gamma(x)g'(x)+\gamma'(x)g(x)\}] \\
&= |\mathbf{W}(x)|\delta_{12}(\nu_0+\nu_2/\mu_2) \\
&\quad +(\nu_2/\mu_2 g(x)^2)[\gamma(x)^2\{\alpha(x)g'(x)+\alpha'(x)g(x)\}-\gamma(x)\beta(x)^2 g'(x) \\
&\quad -2\gamma(x)\beta(x)\beta'(x)g(x)+\beta(x)^2\gamma'(x)g(x)] \\
&= -(g'(x)/g(x)^2)\sigma_1(x)^2|\mathbf{W}(x)|^2(\nu_0+\nu_2/\mu_2) \\
&\quad +\mu_2^4|\mathbf{W}(x)|^2\partial\sigma_1(x)^2(\nu_0+\nu_2/\mu_2)/g(x) \\
&\quad +|\mathbf{W}(x)|^2\sigma_1(x)^2 g'(x)\nu_2/(\mu_2 g(x)^2)-|\mathbf{W}(x)|^2(\nu_2/\mu_2 g(x))\partial\sigma_1(x)^2 \\
&= -\sigma_1(x)^2|\mathbf{W}(x)|^4\nu_0|\mathbf{W}(x)|^2 g'(x)/g(x)^2+|\mathbf{W}(x)|^4\nu_0\partial\sigma_1(x)^2|\mathbf{W}(x)|^2/g(x).
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
nh|\mathbf{W}(x)|^2 c_{14} &= nh|\mathbf{W}(x)|^2 \mathrm{Cov}(\widehat{f}_1(x), \widehat{f}_2'(x)) \\
&\sim |\mathbf{W}(x)|^2[\delta_{11}\delta_{14}v_{11}+\delta_{13}\delta_{14}v_{13}+\delta_{11}\delta_{13}v_{12}/\mu_2+\delta_{12}\delta_{13}v_{22}/\mu_2 \\
&\quad +\delta_{13}^2 v_{14}/\mu_2+\delta_{13}\delta_{14}v_{24}/\mu_2+\delta_{11}\delta_{34}v_{13}+\delta_{13}\delta_{34}v_{33}+\delta_{11}\delta_{33}v_{14}/\mu_2+\delta_{12}\delta_{33}v_{24}/\mu_2 \\
&\quad +\delta_{13}\delta_{33}v_{34}/\mu_2+\delta_{14}\delta_{33}v_{44}/\mu_2] \\
&= |\mathbf{W}(x)|^2[\delta_{11}\delta_{14}\alpha(x)g(x)\nu_0+\delta_{13}\delta_{14}\beta(x)g(x)\nu_0
\end{aligned}
$$

$$
\begin{aligned}
&+\delta_{11}\delta_{13}\partial\{\alpha(x)g(x)\}\nu_2/\mu_2+\delta_{12}\delta_{13}\alpha(x)g(x)\nu_2/\mu_2\\
&+\delta_{13}^2\partial\{\beta(x)g(x)\}\nu_2/\mu_2+\delta_{13}\delta_{14}\beta(x)g(x)\nu_2/\mu_2\\
&+\delta_{11}\delta_{34}\beta(x)g(x)\nu_0+\delta_{13}\delta_{34}\gamma(x)g(x)\nu_0\\
&+\delta_{11}\delta_{33}\partial\{\beta(x)g(x)\}\nu_2/\mu_2+\delta_{12}\delta_{33}\beta(x)g(x)\nu_2/\mu_2\\
&+\delta_{13}\delta_{33}\partial\{\gamma(x)g(x)\}\nu_2/\mu_2+\delta_{14}\delta_{33}\gamma(x)g(x)\nu_2/\mu_2]\\
= & \; |\mathbf{W}(x)|^2\left\{g(x)\nu_0[\delta_{11}\delta_{14}\alpha(x)+\delta_{13}\delta_{14}\beta(x)+\delta_{11}\delta_{34}\beta(x)+\delta_{13}\delta_{34}\gamma(x)]\right.\\
&+[\delta_{11}\delta_{13}\alpha'(x)+\delta_{12}\delta_{13}\alpha(x)+(\delta_{13}^2+\delta_{11}\delta_{33})\beta'(x)+(\delta_{13}\delta_{14}+\delta_{12}\delta_{33})\beta(x)\\
&+\delta_{13}\delta_{33}\gamma'(x)+\delta_{14}\delta_{33}\gamma(x)]g(x)\nu_2/\mu_2\\
&\left.+[\delta_{11}\delta_{13}\alpha(x)+(\delta_{13}^2+\delta_{11}\delta_{33})\beta(x)+\delta_{13}\delta_{33}\gamma(x)]g'(x)\nu_2/\mu_2\right\}\\
= & \; \nu_0|\mathbf{W}(x)|[\delta_{11}\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}\alpha(x)+\delta_{13}\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}\beta(x)\\
&-\delta_{11}\partial\{\sigma_2(x)^2\}\beta(x)-\delta_{13}\partial\{\sigma_2(x)^2\}\gamma(x)]-\delta_{13}\nu_0|\mathbf{W}(x)|g'(x)/g(x)\\
&+g(x)|\mathbf{W}(x)|^2\nu_2/\mu_2[\delta_{11}\delta_{13}\alpha'(x)+(\delta_{13}^2+\delta_{11}\delta_{33})\beta'(x)+\delta_{13}\delta_{33}\gamma'(x)]\\
&-[\delta_{13}\alpha(x)\partial\{\sigma_1(x)^2\}-\delta_{13}\beta(x)\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}+\delta_{33}\beta(x)\partial\{\sigma_1(x)^2\}\\
&-\delta_{33}\gamma(x)\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}]|\mathbf{W}(x)|\nu_2/\mu_2\\
&-\delta_{13}|\mathbf{W}(x)|g'(x)\nu_2/(g(x)\mu_2)\\
&+[\delta_{11}\delta_{13}\alpha(x)+(\delta_{13}^2+\delta_{11}\delta_{33})\beta(x)+\delta_{13}\delta_{33}\gamma(x)]g'(x)|\mathbf{W}(x)|^2\nu_2/\mu_2\\
= & \; \nu_0|\mathbf{W}(x)|/g(x)[\gamma(x)\alpha(x)\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}-\beta(x)^2\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}\\
&-\gamma(x)\beta(x)\partial\{\sigma_2(x)^2\}-\beta(x)\gamma(x)\partial\{\sigma_2(x)^2\}]+\nu_0\beta(x)|\mathbf{W}(x)|g'(x)/g(x)^2\\
&-\nu_2/(g(x)\mu_2)[\gamma(x)\beta(x)\alpha'(x)-(\beta(x)^2+\gamma(x)\alpha(x))\beta'(x)+\beta(x)\alpha(x)\gamma'(x)]\\
&+[\beta(x)\alpha(x)\partial\{\sigma_1(x)^2\}-\beta(x)^2\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}-\alpha(x)\beta(x)\partial\{\sigma_1(x)^2\}\\
&+\alpha(x)\gamma(x)\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}]\nu_2/(g(x)\mu_2|\mathbf{W}(x)|)\\
&+\beta(x)|\mathbf{W}(x)|g'(x)\nu_2/(g(x)^2\mu_2)\\
&-[\gamma(x)\beta(x)\alpha(x)-(\beta(x)^2+\gamma(x)\alpha(x))\beta(x)+\beta(x)\alpha(x)\gamma(x)]g'(x)\nu_2/(g(x)^2\mu_2)\\
= & \; \partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}\nu_0|\mathbf{W}(x)|^2/g(x)+\beta(x)\nu_0|\mathbf{W}(x)|g'(x)/(g(x)^2)\\
&-[\gamma(x)\beta(x)\alpha'(x)-(\beta(x)^2+\gamma(x)\alpha(x))\beta'(x)+\beta(x)\alpha(x)\gamma'(x)]\nu_2/(g(x)\mu_2)\\
&+\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}\nu_2|\mathbf{W}(x)|^2/(g(x)\mu_2)\\
= & \; -\sigma_1(x)\sigma_2(x)\rho(x)\nu_0|\mathbf{W}(x)|^2g'(x)/g(x)^2+\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}\nu_0|\mathbf{W}(x)|^2/g(x).
\end{aligned}
$$

The unweighted case: The estimator in the unweighted case is

$$\tilde{\boldsymbol{\delta}}_x = (\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x)^{-1}\mathbf{X}_x^T\mathbf{K}\otimes\mathbf{I}_2\mathbf{y}$$

so the variance is

$$\begin{aligned}
\mathrm{Var}(\tilde{\boldsymbol{\delta}}_x) &= (\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x)^{-1}\mathbf{X}_x^T\mathbf{K}\otimes I_2\mathbf{V}(x)(\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x)^{-1}\mathbf{X}_x^T\mathbf{K}\otimes I_2 \\
&= \begin{pmatrix} (\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x)^{-1}\mathbf{X}_x^T\mathbf{K}^2\mathbf{D}_{11}\mathbf{X}_x(\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x)^{-1} & (\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x)^{-1}\mathbf{X}_x^T\mathbf{K}^2\mathbf{D}_{12}\mathbf{X}_x(\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x)^{-1} \\ (\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x)^{-1}\mathbf{X}_x^T\mathbf{K}^2\mathbf{D}_{12}\mathbf{X}_x(\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x)^{-1} & (\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x)^{-1}\mathbf{X}_x^T\mathbf{K}^2\mathbf{D}_{22}\mathbf{X}_x(\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x)^{-1} \end{pmatrix}.
\end{aligned}$$

Now,

$$\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x \sim \begin{pmatrix} g(x) & h^2g'(x)\mu_2 \\ h^2g'(x)\mu_2 & h^2g(x)\mu_2 \end{pmatrix}$$

so

$$(\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x)^{-1} \sim \frac{1}{g(x)}\begin{pmatrix} 1 & -g'(x)/g(x) \\ -g'(x)/g(x) & 1/h^2\mu_2 \end{pmatrix}$$

and

$$\mathbf{X}_x^T\mathbf{K}^2\mathbf{D}_{11}\mathbf{X}_x \sim \frac{1}{n}\begin{pmatrix} \frac{1}{h}\sigma_1(x)^2g(x)\nu_0 & h\{\sigma_1(x)^2g'(x)+\partial\sigma_1(x)^2g(x)\}\nu_2 \\ h\{\sigma_1(x)^2g'(x)+\partial\sigma_1(x)^2g(x)\}\nu_2 & h\sigma_1(x)^2g(x)\nu_2 \end{pmatrix}$$

with analogous results when $\mathbf{D}_{11}$ is replaced by $\mathbf{D}_{12}$ or $\mathbf{D}_{22}$. It follows that

$$(\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x)^{-1}\mathbf{X}_x^T\mathbf{K}^2\mathbf{D}_{11}\mathbf{X}_x(\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x)^{-1} \sim$$
$$\frac{1}{nhg(x)}\begin{pmatrix} \sigma_1(x)^2\nu_0 & -g'(x)\sigma_1(x)^2\nu_0/g(x)+\partial\sigma_1(x)^2\nu_2/\mu_2 \\ -g'(x)\sigma_1(x)^2\nu_0/g(x)+\partial\sigma_1(x)^2\nu_2/\mu_2 & \sigma_1(x)^2\nu_2/(h\mu_2^2 \end{pmatrix},$$

$$(\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x)^{-1}\mathbf{X}_x^T\mathbf{K}^2\mathbf{D}_{12}\mathbf{X}_x(\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x)^{-1} \sim$$
$$\frac{1}{nhg(x)}\begin{pmatrix} \sigma_1(x)\sigma_2(x)\rho(x)\nu_0 & -g'(x)\sigma_1(x)\sigma_2(x)\rho(x)\nu_0/g(x) \\ & +\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}\nu_2/\mu_2 \\ -g'(x)\sigma_1(x)\sigma_2(x)\rho(x)\nu_0/g(x) & \sigma_1(x)\sigma_2(x)\rho(x)\nu_2/(h\mu_2)^2 \\ +\partial\{\sigma_1(x)\sigma_2(x)\rho(x)\}\nu_2/\mu_2 & \end{pmatrix},$$

where $\partial f(x) = f'(x)$, and

$$(\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x)^{-1}\mathbf{X}_x^T\mathbf{K}^2\mathbf{D}_{22}\mathbf{X}_x(\mathbf{X}_x^T\mathbf{K}\mathbf{X}_x)^{-1} \sim$$
$$\frac{1}{nhg(x)}\begin{pmatrix} \sigma_2(x)^2\nu_0 & -g'(x)\sigma_2(x)^2\nu_0/g(x)+\partial\sigma_2(x)^2\nu_2/\mu_2 \\ -g'(x)\sigma_1(x)^2\nu_0/g(x)+\partial\sigma_2(x)^2\nu_2/\mu_2 & \sigma_2(x)^2\nu_2/(h\mu_2)^2 \end{pmatrix}.$$

□